XXI. *On Cyclides and Sphero-Quartics.*

By John Casey, *LL.D., M.R.I.A.* *Communicated by* A. Cayley, *F.R.S.*

Received May 11,—Read June 15, 1871.

CHAPTER I.

Section I.—*Spheres cutting orthogonally.*

Art. 1. If $x^2+y^2+z^2+2gx+2fy+2hz+c=0$,

$$x^2+y^2+z^2+2g'x+2f'y+2h'z+c'=0$$

be the equations of two spheres, these spheres will intersect orthogonally if the square of the distance between their centres be equal to the sum of the squares of their radii. Hence we infer that the two spheres whose equations are given above will intersect orthogonally if the condition holds,

$$2ff'+2gg'+2hh'=c+c'. \quad \ldots \ldots \ldots \ldots \quad (1)$$

2. From art. 1 we can easily find the equation of a sphere cutting orthogonally four given spheres, S′, S″, S‴, S⁗. Thus, if the given spheres be

$$x^2+y^2+z^2+2g'x+2f'y+2h'z+c=0 \text{ \&c.},$$

the equation of the orthogonal sphere is

$$\begin{vmatrix} x^2+y^2+z^2, & x, & y, & z, & 1, \\ c', & -g', & -f', & -h', & 1, \\ c'', & -g'', & -f'', & -h'', & 1, \\ c''', & -g''', & -f''', & -h''', & 1, \\ c'''', & -g'''', & -f'''', & -h'''', & 1. \end{vmatrix} \quad \ldots \ldots \ldots \quad (2)$$

Cor. If a sphere S cuts four spheres, S′, S″, S‴, S⁗, orthogonally, it also cuts orthogonally $\lambda S'+\mu S''+\nu S'''+\varrho S''''$ when λ, μ, ν, ϱ are any multiples.

3. The following method finds the equation of the orthogonal sphere in tetrahedral coordinates. Let S′, S″, S‴, S⁗ be the given spheres, then $\lambda S'+\mu S''+\nu S'''+\varrho S''''$ is coorthogonal with S′, S″, &c.; and if the radius of $\lambda S'+\mu S''+\nu S'''+\varrho S''''$ be evanescent, its centre must be a point on the required orthogonal sphere; but if its radius be zero, it represents an imaginary cone and the discriminant vanishes. It is easy to see that λ, μ, ν, ϱ are the tetrahedral coordinates of the centre of $\lambda S'+\mu S''+\nu S'''+\varrho S''''$, the tetrahedron of reference having its angular points at the centres of S′, S″, &c.

Now let the spheres S′, S″, &c. be given in the form

$$(x-a')^2+(y-b')^2+(z-c')^2-r^2=0 \text{ \&c.},$$

and then the required discriminant will be, after dividing by the factor $(\lambda+\mu+\nu+\varrho)^2$*,

$$\begin{aligned}(\lambda+\mu+\nu+\varrho)\{\lambda(a'^2+b'^2+c'^2-r'^2)+\mu(a''^2+b''^2+c''^2-r''^2)\\ +\nu(a'''^2+b'''^2+c'''^2-r'''^2)+\varrho(a''''^2+b''''^2+c''''^2-r''''^2)\}\\ =(a'\lambda+a''\mu+a'''\nu+a''''\varrho)^2+(b'\lambda+b''\mu+b'''\nu+b''''\varrho)^2\\ +(c'\lambda+c''\mu+c'''\nu+c''''\varrho)^2;\end{aligned}$$

and this is readily found to be equivalent to the following equation, in which (S′ S″) &c. denotes the angle of intersection of the spheres S′, S″, &c.:—

$$\left.\begin{aligned}&(\lambda r')^2+(\mu r'')^2+(\nu r''')^2+(\varrho r'''')^2\\ &-2\lambda\mu r'r''\cos(S'S'')-2\lambda\nu r'r'''\cos(S'S''')-2\lambda\varrho r'r''''\cos(S'S'''')\\ &-2\mu\nu r''r'''\cos(S''S''')-2\nu\varrho r'''r''''\cos(S'''S'''')-2\varrho\mu r''''r''\cos(S''''S'')=0.\end{aligned}\right\} \quad . \quad . \quad (3)$$

See my paper "On the Equations of Circles &c.," in the Proceedings of the Royal Irish Academy, vol. ix. pt. iv. p. 410.

This equation is simplified by incorporating the radii r', r'', &c. with the variables $\lambda, \mu, \nu, \varrho$; thus put $\lambda r'=x$, $\mu r''=y$, &c., and we get the equation of the sphere orthogonal to four given spheres in the form

$$\left.\begin{aligned}&x^2+y^2+z^2+w^2-2xy\cos(S'S'')-2xz\cos(S'S''')-2xw\cos(S'S'''')\\ &\qquad-2yz\cos(S''S''')-2zw\cos(S'''S'''')-2wy\cos(S''''S'')=0.\end{aligned}\right\} \quad . \quad . \quad (4)$$

Cor. 1. Hence, if the four given spheres be mutually orthogonal, the equation of their orthogonal sphere in tetrahedral coordinates is

$$(\lambda r')^2+(\mu r'')^2+(\nu r''')^2+(\varrho r'''')^2=0 \text{ or } x^2+y^2+z^2+w^2=0. \quad . \quad . \quad . \quad . \quad (5)$$

Cor. 2. The sphere orthogonal to four given spheres is inscribed in each of the eight quadrics,

$$U^2=(\lambda r'\pm\mu r''\pm\nu r'''\pm\varrho r'''')^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where U denotes the orthogonal sphere†.

* [The vanishing of the factor $(\lambda+\mu+\nu+\varrho)^2$ is the condition that the sphere $\lambda S+\mu S'+\nu S''+\varrho S'''$ may become a plane. Hence $\lambda+\mu+\nu+\varrho=0$ may be regarded as the tangential equation of the centre of the sphere which cuts orthogonally the four spheres S, S′, S″, S‴.—January 1872.]

† [Professor Cayley remarks as follows on this article:—"You give in passing what appears to me an interesting theorem, when you say 'it is easy to see that $\lambda, \mu, \nu, \varrho$ are the tetrahedral coordinates of the centre of the sphere $\lambda S+\mu S'+\nu S''+\varrho S'''=0$.' Take any four quadric surfaces $S=0, S'=0, S''=0, S'''=0$; and establish the relation $\lambda S+\mu S'+\nu S''+\varrho S'''=$ a cone. This establishes between $\lambda, \mu, \nu, \varrho$ and x, y, z, w four lineo-linear equations, so that, eliminating either set of variables, we have between the other set a quartic equation; moreover, the variables of each set are proportional to cubic functions of the other set (see my "Memoir on Quartic Surfaces," vol. iii. pp. 19–69 of the Proceedings of the London Mathematical Society). Then your theorem is, that when the four quadrics have a common conic, the x, y, z, w and $\lambda, \mu, \nu, \varrho$ are linear functions each of the other, so that the two quartic surfaces are homographically related, or, by a proper interpretation of the coordinates, may be regarded as being one and the same surface." My theorem, that $\lambda, \mu, \nu, \varrho$ are the tetrahedral coordinates of the centre of the sphere $\lambda S+\mu S'+\nu S''+\varrho S'''=0$, is easily proved as follows: the centre of the sphere $\lambda S+\mu S'+\nu S''+\varrho S'''$ is evidently the mean centre of the centres of S, S′, S″, S‴ for the system of multiples $\lambda, \mu, \nu, \varrho$; in other words, it is the centre of gravity of four masses proportional to $\lambda, \mu, \nu, \varrho$ placed at those points. Hence the proposition follows at once by a well-known theorem in Statics.—January 1872.]

4. If $W \equiv (a, b, c, d, l, m, n, p, q, r \between \alpha, \beta, \gamma, \delta)^2 = 0$, where $\alpha=0, \beta=0, \gamma=0, \delta=0$ are the equations of four given spheres, which I shall by analogy to known systems call the spheres of *reference*, then $W=0$ is evidently the most general equation of a surface of the fourth degree, having the imaginary circle at infinity as a double line. Such a surface has been called by MOUTARD an "anallagmatic surface," and by DARBOUX a "cyclide" (see 'Comptes Rendus,' June 7, 1869). I shall adopt the latter name.

5. The cyclide $W=0$ is by the usual theory the envelope of the sphere

$$x\alpha+y\beta+z\gamma+w\delta=0,$$

where x, y, z, w are variable multiples, provided the condition holds:

$$\begin{vmatrix} a, & n, & m, & p, & x, \\ n, & b, & l, & q, & y, \\ m, & l, & c, & r, & z, \\ p, & q, & r, & d, & w, \\ x, & y, & z, & w, & 0; \end{vmatrix} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

now the sphere $x\alpha+y\beta+z\gamma+w\delta=0$ cuts orthogonally the Jacobian of α, β, γ, δ, and the equation (7) is the equation of a quadric. Hence we have the following theorem:—

A quartic cyclide is the envelope of a variable sphere whose centre moves on a given quadric, and which cuts a given fixed sphere orthogonally.

6. If the equation of the cyclide be of the form

$$(a, b, c, f, g, h \between \alpha, \beta, \gamma)^2 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

it is shown, as in the last article, that it is the envelope of a variable sphere whose centre moves along a plane conic and which cuts a given fixed sphere orthogonally. Now from the form of equation (8) it is evident that this species of cyclide has two nodes, namely, the two points common to the three spheres of reference α, β, γ, and that these nodes are conic nodes, that is, nodes which have these points as vertices of tangent cones to the cyclide. I shall call this species of cyclide a *binodal* cyclide*.

SECTION II.—*Generalization of methods of Section* I.

7. The results of Section I. admit of important generalization, to the exposition of which I shall devote a few articles.

Let $S^{\frac{1}{2}}-A=0$, $S^{\frac{1}{2}}-B=0$ be two quadrics inscribed in the same quadric, $S \equiv x^2+y^2+z^2+w^2$, A and B being the planes $ax+a'y+a''z+a'''w=0$ and $bx+b'y+b''z+b'''w=0$ respectively; we see that $S^{\frac{1}{2}}-A+k(S^{\frac{1}{2}}-B)$ is the equation of a quadric inscribed in S, and passing through one of the conics of intersection of $S-A^2$ and $S-B^2$, namely, through the common intersection of these two quadrics with the plane $A-B=0$. But if we clear $(S^{\frac{1}{2}}-A)+k(S^{\frac{1}{2}}-B)=0$ from radicals, the discriminant

* [The cyclide (8) must, from the form of the equation, have two nodes; but in certain special cases which will be discussed in the sequel, it will have one or two additional nodes.—January 1872.]

of the result equated to zero gives, as is easily seen,

$$(1-S'')k^2+2(1-R)k+(1-S')=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where S′, S″ are the results of substituting the coordinates of the poles of the planes A and B in S, and R the result of substituting the coordinates of the pole of A in B.

We should arrive at the same result if we had taken the equations of the two quadrics under the form $S^{\frac{1}{2}}+A$ and $S^{\frac{1}{2}}+B$. But if we had worked with $S^{\frac{1}{2}}\mp A$ and $S^{\frac{1}{2}}\pm B$, we should get

$$(1-S'')k^2+2(1+R)k+(1-S'). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

8. As the equations (9) and (10) are of the second degree in k, we see that, through each conic of intersection of $S-A^2$ and $S-B^2$, there pass two cones circumscribed to S. The equations of these cones are obtained by eliminating k between $S^{\frac{1}{2}}-A+k(S^{\frac{1}{2}}-B)$, and the two equations (9) and (10). They are:

$$(1-S'')(S^{\frac{1}{2}}-A)^2-2(1-R)(S^{\frac{1}{2}}-A)(S^{\frac{1}{2}}-B)+(1-S')(S^{\frac{1}{2}}-B)^2, \quad . \quad . \quad (11)$$

$$(1-S'')(S^{\frac{1}{2}}-A)^2-2(1+R)(S^{\frac{1}{2}}-A)(S^{\frac{1}{2}}+B)+(1-S')(S^{\frac{1}{2}}+B)^2. \quad . \quad . \quad (12)$$

These cones correspond to the limiting points of two spheres, as these latter are evidently imaginary cones passing through the circle of intersection of the two spheres and circumscribed to the imaginary circle at infinity.

9. If we put

$$1-R=\sqrt{(1-S')(1-S'')}\cos\theta,$$

$$1+R=\sqrt{(1-S')(1-S'')}\cos\phi,$$

the ratio of the roots of equation (9) is $e^{2\theta\sqrt{-1}}$, and of equation (10) $e^{2\phi\sqrt{-1}}$. Now if $\theta=\frac{\pi}{2}$ the ratio of the roots is negative unity, and we have an harmonic pencil of four planes, namely the planes A, B, and the planes passing through the intersection of the planes A, B, and which are also planes of contact of the cones of article (8) with the quadric (S); in other words, the poles of A and B and the vertices of the cones form an harmonic range of points. When two quadrics, then, are connected by the relation

$$1\pm R=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

I shall, by an extension of a known term, say that they cut *orthogonally* or *harmonically*.

10. It is easy to see that, being given by its general equation, a quadric S, and two planes $\lambda x+\mu y+\nu z+\varrho w$, $\lambda' x+\mu' y+\nu' z+\varrho' w$, the result of substituting the coordinates of the pole with respect to the quadric of one of the planes in the equation of the other, multiplied by the discriminant of the quadric, is equal to the determinant:

$$\begin{vmatrix} a, & n, & m, & p, & \lambda, \\ n, & b, & l, & q, & \mu, \\ m, & l, & c, & r, & \nu, \\ p, & q, & r, & d, & \varrho, \\ \lambda, & \mu', & \nu', & \varrho', & 0; \end{vmatrix}$$

and denoting this by Π, we infer from equation (13) that the condition that should cut orthogonally two quadrics, $S-(\lambda x+\mu y+\nu z+\varrho w)^2$, $S-(\lambda' x+\mu' y+\nu' z+\varrho' w)^2$, both inscribed in the same quadric S given by its general equation, is the invariant relation

$$\Delta \pm \Pi = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

11. To find the equation of a quadric cutting four given quadrics orthogonally. Let

$$S^{\frac{1}{2}} \pm A, \quad S^{\frac{1}{2}} \pm B, \quad S^{\frac{1}{2}} \pm C, \quad S^{\frac{1}{2}} \pm D$$

be the four given quadrics. It follows from equation (13) that we must have

$$a\lambda + a'\mu + a''\nu + a'''\varrho \pm 1 = 0,$$
$$b\lambda + b'\mu + b''\nu + b'''\varrho \pm 1 = 0,$$
$$c\lambda + c'\mu + c''\nu + c'''\varrho \pm 1 = 0,$$
$$d\lambda + d'\mu + d''\nu + d'''\varrho \pm 1 = 0,$$

λ, μ, ν, ϱ being the coordinates of the pole of the plane of contact of the sought quadric with respect to S, and that this quadric will be then $S^{\frac{1}{2}} - (\lambda x + \mu y + \nu z + \varrho w) = 0$.

Hence, eliminating λ, μ, ν, ϱ from these five equations, we get

$$\begin{vmatrix} S^{\frac{1}{2}}, & x, & y, & z, & w, \\ \mp 1, & a, & a', & a'', & a''', \\ \mp 1, & b, & b', & b'', & b''', \\ \mp 1, & c, & c', & c'', & c''', \\ \mp 1, & d, & d', & d'', & d''', \end{vmatrix} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

where the double signs of the first column answer to those of the binomial, $S^{\frac{1}{2}} \pm A$. Hence if we denote for shortness by the notation $(S^{\frac{1}{2}}\ a\ b'\ c''\ d''')$ the determinant (16), in the case where all the units in the first column are positive, we shall have eight orthogonal quadrics, whose equations are as follows:—

$$(S^{\frac{1}{2}} \quad a \quad b' \quad c'' \quad d''') = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$
$$(S^{\frac{1}{2}} \quad -a \quad b' \quad c'' \quad d''') = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$
$$(S^{\frac{1}{2}} \quad a \quad -b' \quad c'' \quad d''') = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$
$$(S^{\frac{1}{2}} \quad a \quad b' \quad -c'' \quad d''') = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$
$$(S^{\frac{1}{2}} \quad a \quad b' \quad c'' \quad -d''') = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$
$$(S^{\frac{1}{2}} \quad -a \quad -b' \quad c'' \quad d''') = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$
$$(S^{\frac{1}{2}} \quad -a \quad b' \quad -c'' \quad d''') = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$
$$(S^{\frac{1}{2}} \quad -a \quad b' \quad c'' \quad -d''') = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Thus, for example, equation (20) developed is

$$\begin{vmatrix} S^{\frac{1}{2}}, & x, & y, & z, & w, \\ 1, & a, & a', & a'', & a''', \\ 1, & b, & b', & b'', & b''', \\ 1, & -c, & -c', & -c'', & -c''', \\ 1, & d, & d', & d'', & d''', \end{vmatrix} = 0.$$

12. Denoting by $J_1 \ldots J_8$ the eight orthogonal quadrics (17) ... (24), and remembering that λ, μ, ν, ϱ are the coordinates of the pole of the plane of contact of one of these surfaces with S, since these coordinates satisfy the first four equations of art. 11, we see easily that they belong to the point common to the system of six planes represented by the system of six equations,

$$\pm A = \pm B = \pm C = \pm D,$$

in which the arrangement of the signs correspond to the quadric which we consider. We have then the following theorem:—

The poles of contact of the eight orthogonal quadrics $J_1 \ldots J_8$ *are the eight radical centres of the four quadrics,* $S-A^2$, $S-B^2$, $S-C^2$, $S-D^2$.

13. The polar of the point λ, μ, ν, ϱ, with respect to $S-A^2$, is

$$\lambda(x-Aa)+\mu(y-Aa')+\nu(zAa'')+\varrho(w-Aa'''),$$

and this reduces, in virtue of the first equation of art. 11, to

$$\lambda x+\mu y+\nu z+\varrho w=\mp A;$$

and eliminating λ, μ, ν, ϱ from this and the four equations of the same article, we get

$$\begin{vmatrix} \pm A, & x, & y, & z, & w, \\ \pm 1, & a, & a', & a'', & a''', \\ \pm 1, & b, & b', & b'', & b''', \\ \pm 1, & c, & c', & c'', & c''', \\ \pm 1, & d, & d', & d', & d''', \end{vmatrix} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

where the choice of signs depends on the quadric J. This is evidently the plane of contact of one of the conics of intersection of J and $S-A^2$. We have then the following construction for the eight orthogonal quadrics:—

Let us imagine tangent cones whose vertices are the eight radical centres, and the required quadrics pass through the conics of contact.

14. The equations of J_1, J_2, &c. take a very simple form when referred to the tetrahedron which has for vertices the poles of the planes A, B, C, D with respect to S in tetrahedral coordinates.

Let x', y', z', w' be the new coordinates of the point (x, y, z, w), the poles being a, a', a'',

a''', b, b', b'', b''', &c., and we have then the following substitutions to make·

$$\begin{aligned} x &= ax' + by' + cz' + dw' , \\ y &= a'x' + b'y' + c'z' + d'w' , \\ z &= a''x' + b''y' + c''z' + d''w' , \\ w &= a'''x' + b'''y' + c'''z' + d'''w' ; \end{aligned}$$

and hence, by the equation of art. 11,

$$\lambda x + \mu y + \nu z + \varrho w = \pm x' \pm y' \pm z' + w'.$$

Consequently the transformed equation of J (corresponding to the choice of the double signs) is simply

$$\pm(x' \pm y' \pm z' \pm w') = S^{\frac{1}{2}}.$$

Hence

$$\begin{aligned} (\pm x' \pm y' \pm z' \pm w')^2 &= (ax' + by' + cz' + dw')^2 + (a'x' + b'y' + c'z' + d'w')^2 \\ &+ (a''x' + b''y' + c''z' + d''w')^2 + (a'''x' + b'''y' + c'''z' + d'''w')^2. \end{aligned}$$

This can be written in a more convenient manner by the following substitution, and by suppressing accents as being no longer necessary.

Let us denote the result of substituting the coordinates of the pole of

B	in	C	by	L	and of	A	in	D	by	P,
C	,,	A	,,	M	,,	B	,,	D	,,	Q,
A	,,	B	,,	N	,,	C	,,	D	,,	R,

and we shall have the equation of J_1 in the following form:

$$\left.\begin{aligned} J_1 \equiv &(1-S')x^2 + (1-S'')y^2 + (1-S''')z^2 + (1-S'''')w^2 \\ &+ 2(1-L)yz + 2(1-M)zx + 2(1-N)xy \\ &+ 2(1-P)xw + 2(1-Q)yw + 2(1-R)zw = 0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (26)$$

15. The equation J_1 is the locus of all the double points of the quadric

$$\lambda(S^{\frac{1}{2}} - A) + \mu(S^{\frac{1}{2}} - B) + \nu(S^{\frac{1}{2}} - C) + \varrho(S^{\frac{1}{2}} - D).$$

In fact this is equivalent to the equation

$$(\lambda + \mu + \nu + \varrho)^2 S - (\lambda A + \mu B + \nu C + \varrho D)^2,$$

the discriminant of which is easily found to be

$$\begin{aligned} (\lambda + \mu + \nu + \varrho)^2 &= (\lambda a' + \mu b + \nu c + \varrho d)^2 + (\lambda a' + \mu b' + \nu c' + \varrho d') \\ &+ (\lambda a'' + \mu b'' + \nu c'' + \varrho d'')^2 + (\lambda a''' + \mu b''' + \nu c''' + \varrho d''')^2 ; \end{aligned}$$

and λ, μ, ν, ϱ being replaced by x, y, z, w, we have the equation of J_1.—Q.E.D.

It is instructive to compare the modes of investigation employed in this article and article 3 of the last section.

16. The equation (26) of J_1 can be written in a more suitable form by means of the anharmonic angles of art. 9; for this purpose let us denote

$$\frac{1-R}{\sqrt{(1-S')(1-S'')}} \text{ by } \cos L; \quad \frac{1+R}{\sqrt{(1-S')(1-S'')}} \text{ by } -\cos L', \text{ \&c.}$$

Let us denote also S' by $\cos^2 \varrho'$, S'' by $\cos^2 \varrho''$, &c. It is evident that the angles L, L', ϱ', ϱ'', &c. may be either real or imaginary; when the substitutes are made, we get

$$\left.\begin{aligned} J_1 \equiv & x^2 \sin^2 \varrho' + y^2 \sin^2 \varrho'' + z^2 \sin^2 \varrho''' + w^2 \sin^2 \varrho'''' \\ & + 2yz \sin \varrho'' \sin \varrho''' \cos L + 2zx \sin \varrho''' \sin \varrho' \cos M + 2xy \sin \varrho' \sin \varrho'' \cos N \\ & + 2xw \sin \varrho' \sin \varrho'''' \cos P + 2yw \sin \varrho'' \sin \varrho'''' \cos Q + 2zw \sin \varrho''' \sin \varrho'''' \cos R = 0. \end{aligned}\right\} \quad (27)$$

Compare equation (3), art. 3.

Cor. If the four quadrics $S^{\frac{1}{2}}-A$, $S^{\frac{1}{2}}-B$, &c. be mutually orthogonal, the equation of their orthogonal quadric will be $J_1 \equiv x^2 \sin^2 \varrho' + y^2 \sin^2 \varrho'' + z^2 \sin^2 \varrho''' + w^2 \sin^2 \varrho''''$, or of the form $x^2+y^2+z^2+w^2=0$, and there will be only one orthogonal quadric instead of eight.

Observation. This section is abridged from a Memoir by me in TORTOLINI's 'Annali di Matematica,' serie ii. tomo ii. fasc. 4.

CHAPTER II.

SECTION I.—*Generation of Cyclides.*

17. We have seen that a cyclide is the envelope of a variable sphere whose centre moves along a given quadric, and which cuts a given sphere orthogonally (see art. 5). I shall call the variable sphere the *generating sphere* (a name which I find more convenient than *enveloped sphere* or enveloppée), and the quadric which is the locus of the centres of the generating sphere I shall call the *focal quadric*, a term expressive of an important property which it possesses. In DE LA GOURNERIE's Memoir "Sur les lignes Sphériques," he uses the name *déférente* in an analogous case (see LIOUVILLE's Journal, 1869). The sphere which is cut orthogonally we shall call the sphere of inversion, and it will be always denoted by the letter U, unless the contrary be stated, and the focal quadric by the letter F.

18. If we draw any tangent plane to F this will intersect F in two lines, the generating lines of F at the point of contact. Now let us conceive three spheres whose centres are at the intersections of these lines and at a consecutive point on each line respectively; then if they cut U orthogonally, the two points common to the three will evidently be their points of contact with their envelope. Now it is evident that these points are the limiting points of the system composed of U, and the tangent plane to F. Hence we have the following method of generating cyclides:—

Being given a quadric F *and a sphere* U, *draw any tangent plane* P *to* F, *the locus of the limiting points of* U *and* P *will be a cyclide.*

19. Let the two lines in which the tangent plane P of the last article cuts F be denoted by L, L′; now all the generating spheres whose centres are on L have a common circle of intersection; if this circle be called C, and the corresponding circle for L′ be called C′, then these circles are evidently homospheric, and the two points common to them are plainly points on the cyclide. I say moreover that the circles C, C′ lie altogether on the cyclide. For through L draw another tangent plane to F intersecting F in another line L″; then corresponding to this line we have another circle, C″, which is also homospheric with C, and their points of intersection are points on the cyclide; hence the circle C lies altogether in the cyclide, and so do the circles C′, C″, &c. Hence we infer the following method of generating cyclides analogous to the rectilinear generation of quadrics:—

Being given three circles, C, C′, C″, *cutting* U *orthogonally, the intersection of their planes being the centre of* U, *then the envelope of a variable circle whose motion is directed by cutting each of these circles twice is a cyclide.*

Cor. 1. Every generating sphere of a cyclide intersects it in the two generating circles passing through its points of contact with the cyclide.

Cor. 2. The generating spheres touch but do not intersect the cyclide if their focal quadric be not a ruled surface.

20. If the sphere U reduce to a point, which will happen when the four spheres of reference α, β, γ, δ (see art. 4) pass through a common point, the method of generating cyclides given in art. 18 becomes simplified as follows:—

Being given a quadric F *and a point* U, *then the locus of the reflection of* U *made by any tangent plane to* F *will be a cyclide.* This is plainly equivalent to the following:— *The pedal of a quadric is a cyclide;* or again, *the inverse of a quadric with respect to any arbitrary point is a cyclide.*

Or we may state the whole matter thus:—Being given a quadric F and a point U, from U draw a perpendicular UT on any tangent plane to F, and on UT take P, P′ in opposite directions such that $UT^2-TP^2=UT^2-TP'^2=k^2$, where k is a constant, then the locus of P, P′ is a cyclide.

There are three cases to be considered.

1°. If k^2 be positive the sphere U is real.

2°. If k^2 be negative the sphere U is imaginary; this will happen when the radical centre of the spheres of reference, α, β, γ, δ, is internal to these spheres.

3°. If k^2 vanish, U reduces to a point. The cyclide is in this case the inverse of a quadric.

The point U is a nodal point on the cyclide. The tangent planes to the cyclide at the node U form a cone, which is reciprocal to the cone whose vertex is at U and which circumscribes F. Hence the point U will be a conic node when F is either an ellipsoid or hyperboloid. We shall examine more minutely the species of the node in this case when we come to the Chapter on the Inversion of Cyclides.

21. If the focal quadric F be a cone the cyclide becomes modified in a remarkable way,

which it is necessary to examine, as this species of cyclide will occupy much of our space in the present memoir.

Since all the tangent planes of a cone pass through the same point, and since every tangent plane determines two points on the cyclide, it is plain that all the points lie on the surface of a sphere whose centre is at the vertex of the cone.

Again, since the cone is a ruled surface, each edge of it will determine, as in art. 19, a circle which will be a generating circle of the cyclide; but the circle will not in this case lie altogether in the envelope as in art. 19, because in art. 19 the points of contact of any line on the quadric are distinct for all the planes passing through it, whereas in the cone only one tangent plane, properly so called, can be drawn through any edge of it. But although the circles which answer to each edge of the cone do not lie altogether in the cyclide, yet the envelope of these circles is the cyclide, which in this case is evidently a twisted curve, which, as will be shown, is of the fourth degree. On this account I have called this species of cyclide, for the sake of distinction, a sphero-quartic.

22. Since the planes of the generating circles in the last article are perpendicular to the edges of the focal cone, the envelope of these planes is another cone; and as each plane passes through the centre of the sphere U, the vertex of the second cone is at the centre of U. Hence the sphero-quartic is the intersection of a sphere and a cone. Hence we have the following theorem:—*When a cyclide reduces to a sphero-quartic, it is the intersection of a sphere and a quadric.*

23. If we denote the sphere on which we have proved the sphero-quartic lies by Ω, then the generating circles are circles on Ω; and as Ω evidently cuts U orthogonally, the circle of intersection of U and Ω, which we denote by J, will be orthogonal to all the generating circles, and the focal cone pierces Ω in a sphero-conic. Hence we have the following method of generating sphero-quartics:—

Being given a circle J *on a sphere, and a sphero-conic on the same sphere. A sphero-quartic is the envelope of a variable circle whose centre moves along the sphero-conic, and which cuts the circle* J *orthogonally.*

24. From the last article we infer this other method of generating sphero-quartics.

Let F be a sphero-conic on a sphere Ω, U a point on the surface of Ω; from U draw an arc UT perpendicular to any great circle tangential to F, and take two points, P, P′, such that

$$\cos \mathrm{UT} : \cos \mathrm{TP} = \cos \mathrm{UT} : \cos \mathrm{TP}' = k,$$

where k is a constant.

The locus of the points P, P′ is a sphero-quartic.

Cor. If $k=1$ the point P coincides with U, and the point will in this case be a double point in the sphero-quartic, and the sphero-quartic itself will be the inverse of a plane conic from a point outside the plane of the conic. In fact if the sphere Ω be inverted into a plane from the point U, it is easy to see that the point P′ will be inverted into a point whose locus is a conic.

SECTION II.—*Generation of Quartic Surfaces having a Conic Nodal Line.*

Lemma. If $S = x^2 + y^2 + z^2 + w^2 = 0$, $A = ax + by + cz + dw = 0$, then the result of the operation $\lambda\frac{d}{dx} + \mu\frac{d}{dy} + \nu\frac{d}{dz} + \varrho\frac{d}{dw}$ performed on the quadric $S^{\frac{1}{2}} - A$ is

$$(a\lambda + b\mu + c\nu + d\varrho)S^{\frac{1}{2}} - (\lambda x + \mu y + \nu z + \varrho w),$$

which is evidently connected with $S^{\frac{1}{2}} - A$ by the invariant relation (13) of art 9. Hence we have the following theorem:—

The result of the operation $\lambda\frac{d}{dx} + \mu\frac{d}{dy} + \nu\frac{d}{dz} + \varrho\frac{d}{dw}$ *performed on a quadric of the form* $S^{\frac{1}{2}} \pm A$ *is a quadric orthogonal to* $S^{\frac{1}{2}} \pm A$.

25. Being given quadric J inscribed in another S and a point $(\alpha, \beta, \gamma, \delta)$, we can find, by the method of the preceding lemma, a quadric Q orthogonal to J, whose pole of contact with S is the point $(\alpha, \beta, \gamma, \delta)$. When $(\alpha, \beta, \gamma, \delta)$ varies, Q varies also; and I say if the locus of $(\alpha, \beta, \gamma, \delta)$ is a quadric F, that the envelope of Q is a quartic surface having a conic nodal line.

Demonstration. Let $Q_1 = S^{\frac{1}{2}} - A$, $Q_2 = S^{\frac{1}{2}} - B$, $Q_3 = S^{\frac{1}{2}} - C$, $Q_4 = S^{\frac{1}{2}} - D$ be four particular quadrics of the system cutting J orthogonally. Let us consider the quadric Q as having its pole of contact with S at the centre of mean distances of the poles of contact of Q_1, Q_2, Q_3, Q_4 for any suitable system of multiples x, y, z, w, and as we are only concerned with their mutual ratios, we can put $x + y + z + w = 1$. Hence we get the following system of eqnations:—

$$\begin{aligned}
\alpha &= ax + by + cz + dw,\\
\beta &= a'x + b'y + c'z + d'w,\\
\gamma &= a''x + b''y + c''z + d''w,\\
\delta &= a'''x + b'''y + c'''z + d'''w;
\end{aligned}$$

and from these we have $Q = Q_1x + Q_2y + Q_3z + Q_4w$; and since the locus of $(\alpha, \beta, \gamma, \delta)$ is the quadric F, we see by substitution of the preceding values of $\alpha, \beta, \gamma, \delta$ in its equation, that x, y, z, w are connected by an equation of the second degree. If we denote this equation by

$$(a, b, c, d, l, m, n, p, q, r \between x, y, z, w)^2 = 0,$$

we have to find the envelope of $Q_1x + Q_2y + Q_3z + Q_4w$ subject to this condition. The theory of envelopes gives

$$\begin{vmatrix}
a, & n, & m, & p, & Q_1,\\
n, & b, & l, & q, & Q_2,\\
m, & l, & c, & r, & Q_3,\\
p, & q, & r, & d, & Q_4,\\
Q_4, & Q_2, & Q_3, & Q_4, & 0,
\end{vmatrix} = 0.$$

Hence the required envelope is

$$\begin{vmatrix} a\ , & n\ , & m\ , & p\ , & S^{\frac{1}{2}}-A, \\ n\ , & b\ , & l\ , & q\ , & S^{\frac{1}{2}}-B, \\ m\ , & l\ , & c\ , & r\ , & S^{\frac{1}{2}}-C, \\ p\ , & q\ , & r\ , & d\ , & S^{\frac{1}{2}}-D, \\ S^{\frac{1}{2}}-A, & S^{\frac{1}{2}}-B, & S^{\frac{1}{2}}-C, & S^{\frac{1}{2}}-D, & 0, \end{vmatrix}=0. \quad . \ . \ . \ (28)$$

The expansion of this determinant gives evidently a result of the form $U_2+U_1S^{\frac{1}{2}}=0$, where U_2 represents a function of the second degree, and U_1 a function of the first degree in the variables; and clearing off radicals we get $U_2^2-U_1^2S=0$; and this is the equation of a surface of the fourth degree, having the conic of intersection of U_2 and U_1 as a double line. Hence the proposition is proved.

26. The quantities x, y, z, w of the last article are evidently proportional to the tetrahedral coordinates of the point $\alpha, \beta, \gamma, \delta$, referred to the tetrahedron whose vertices are the poles of the planes A, B, C, D of the quadrics $S-A^2$, $S-B^2$, $S-C^2$, $S-D^2$, so that the equation of condition in x, y, z, w is only the equation of the surface F referred to this tetrahedron. Hence the method of generation of surfaces of the fourth degree having a conic for a nodal line is exactly the same as the method of generating cyclides given in art. 5; and in fact the two surfaces are identical, since the cyclide has the imaginary circle at infinity for a nodal line, so that by linear transformation we could get one surface from the other; and to every property of a cyclide there is a corresponding property of the more general surface here considered: but I thought that it would be useful to show that their equations, the equations of the surface cutting them orthogonally &c., are identical in form; so that for every theorem which I shall prove to hold for a cyclide the reader may if he chooses put in the more general surface here considered*.

* [Professor CAYLEY has remarked to me that, instead of the method of Chapter II., the immediate generalization would be to consider, instead of spheres, quadric surfaces of the form $S+LM$, $S+LN$, &c., and that it is a *further* generalization, or rather an extension, of $S-A^2$, $S-B^2$, &c. Professor CAYLEY remarks that it is a pity to omit the intermediate step. Before Professor CAYLEY had drawn my attention to it, the intermediate step had not occurred to me; however, any person who reads Chapter II. will find it easy to supply, by the assistance of the two following propositions, the omissions referred to:—

I. $S+LM=0$, $S+LN=0$ are two quadrics; it is required to find the condition that the pole of L with respect to $S+LM$ will be the pole of $M-N$ with respect to $S+LN$. Let

$$S+LM=x^2+y^2+z^2+w^2+2(by+cz+dw)x,$$

$$S+LN=x^2+y^2+z^2+w^2+2(b'y+c'z+d'w)x;$$

and let λ, μ, ν, ρ be the coordinates of the pole of the plane x with respect to the quadric $S+LM=0$; then we get the four equations:

$$\lambda+b\mu+c\nu+d\rho=1, \qquad \lambda b+\mu=0,$$

$$\lambda c+\nu=0, \qquad \lambda d+\rho=0.$$

CHAPTER III.

SECTION I.—*Different forms of the Equations of Cyclides.*

27. Let T be a tangent plane to the focal quadric F; P, P′ the corresponding points of the cyclide; then (art. 18) P, P′ are the limiting points of the sphere U and the plane T.

Hence

$$\lambda=\frac{1}{1-(b^2+c^2+d^2)},\quad \mu=\frac{-b}{1-(b^2+c^2+d^2)},$$

$$\nu=\frac{-c}{1-(b^2+c^2+d^2)},\quad \varrho=\frac{-d}{1-(b^2+c^2+d^2)}.$$

Hence forming the condition that the polar plane of the point $(\lambda, \mu, \nu, \varrho)$ with respect to the quadric S+LN is

$$(b-b')y+(c-c')z+(d-d')w=0,$$

we get

$$bb'+cc'+dd'=1.$$

This condition I propose to call the orthotomic invariant of the two quadrics.

If we take the more general forms,

$$x^2+y^2+z^2+w^2+2(ax+by+cz+dw)x,$$

$$x^2+y^2+z^2+w^2+2(a'x+b'y+c'z+d'w)x,$$

for S+LM, S+LN, these may, without loss of generality, be written in the more compact forms

$$ax^2+y^2+z^2+w^2+2(by+cz+dw)x,$$

$$a'x^2+y^2+z^2+w^2+2(b'y+c'z+d'w)x;$$

and we find, as before, the orthotomic invariant to be

$$2bb'+2cc'+2dd'=a+a'.$$

Compare equation (1), article 1.

The two quadrics related, as here considered, have many important properties. Thus *the poles of the plane* L *with respect to the quadrics, and the four points in which the line of connexion of these poles meets the quadrics, form a system of six points in involution.*

Def. Two quadrics related as in this proposition may be said to cut orthogonally.

II. Given five quadrics, S+LM, S+LN, &c., where

$$M\equiv a'x+b'y+c'z+d'w=0,$$

$$N\equiv a''x+b''y+c''z+d''w=0,$$

then the condition that the five quadrics should be coorthogonal is the determinant

$$\begin{vmatrix} a', & b', & c', & d', & 1, \\ a'', & b'', & c'', & d'', & 1, \\ a''', & b''', & c''', & d''', & 1, \\ a'''', & b'''', & c'''', & d'''', & 1, \\ a''''', & b''''', & c''''', & d''''', & 1, \end{vmatrix}=0.$$

Hence we infer the following theorem:—If α, β, γ, δ be any four quadrics of the form S+LM=0, S+LN=0, &c., then the quadric $\lambda\alpha+\mu\beta+\nu\gamma+\varrho\delta$ is coorthogonal with α, β, γ, δ, and the pole of the plane L=0 with respect to $\lambda\alpha+\mu\beta+\nu\gamma+\rho\delta$ will be a point whose tetrahedral coordinates are proportional to λ, μ, ν, ϱ, the angular points of the tetrahedron of reference being the poles of L with respect to α, β, γ, δ respectively.—January 1872.]

Let a, b, c, d be the centres of the spheres of reference, α, β, γ, δ; r', r'', r''', r'''' their radii; then, since U is the Jacobian of α, β, γ, δ, the tangents from a, b, c, d to U are equal to r', r'', r''', r'''' respectively. Again, let perpendiculars from a, b, c, d on the tangent plane T be denoted by λ, μ, ν, ϱ. Now the result of substituting the coordinates of P in $\alpha = Pa^2 - r^2 =$ difference of squares of tangents from a to the limiting point P, and the orthogonal sphere $U = 2\lambda . OP$ (O being the centre of U). Hence the results of substituting the coordinates of any point P of the cyclide in the equations of the spheres of reference are proportional to the perpendiculars λ, μ, ν, ϱ. Hence we have the following theorem:—

If $(a, b, c, d, l, m, n, p, q, r)(\alpha, \beta, \gamma, \delta)^2 = 0$ *be the equation of any cyclide,*

$$(a, b, c, d, l, m, n, p, q, r)(\lambda, \mu, \nu, \varrho)^2 = 0$$

is the tangential equation of the corresponding focal quadric of the cyclide.

Cor. 1. Hence if we are given the equation of the focal quadric, we are given the equation of the cyclide, and *vice versâ.*

Cor. 2. Hence, when the sphere of inversion U and the focal quadric F of a cyclide are given, the cyclide is determined; but U is determined by four constants and F by nine. Hence a cyclide is determined by $4+9=13$ constants.

28. By means of the last article we are enabled to get a very important expression for the sphere U in terms of the four spheres of references. Thus, since a cyclide is the envelope of a variable sphere cutting U orthogonally, and whose centre moves along the surface of a given quadric F, now if the given quadric F be the sphere U itself, it is plain that the cyclide will in this case be the sphere U counted twice, that is U^2. But the equation of U in tetrahedral coordinates, x, y, z, w being the coordinates, is (see art. 3)

$$\begin{gathered}(r'x)^2+(r''y)^2+(r'''z)^2+(r''''w)^2\\ -2r'r''xy\cos(\alpha\beta)-2r'r'''xz\cos(\alpha\gamma)-2r'r''''xw\cos(\alpha\delta)\\ -2r''r'''yz\cos(\beta\gamma)-2r''r''''yw\cos(\beta\delta)-2r'''r''''zw\cos(\gamma\delta)=0.\end{gathered}$$

Hence, forming the corresponding tangential equation, and substituting α, β, γ, δ for the variables, we get the following determinant for the square of U:—

$$\begin{vmatrix} -r'^2, & r'r''\cos(\alpha\beta), & r'r'''\cos(\alpha\gamma), & r'r''''\cos(\alpha\delta), & \alpha, \\ r''r'\cos(\beta\alpha), & -r''^2, & r''r'''\cos(\beta\gamma), & r''r''''\cos(\beta\delta), & \beta, \\ r'''r'\cos(\gamma\alpha), & r'''r''\cos(\gamma\beta), & -r'''^2, & r'''r''''\cos(\gamma\delta), & \gamma, \\ r''''r'\cos(\delta\alpha), & r''''r''\cos(\delta\beta), & r''''r'''\cos(\delta\gamma), & -r''''^2, & \delta, \\ \alpha, & \beta, & \gamma, & \delta, & 0. \end{vmatrix}$$

This determinant may be simplified as follows:—divide the first row by r', the second

by r'', &c. Again divide the first column by r', the second by r'' &c., and we get the following result:—

$$U^2 = \begin{vmatrix} -1, & \cos(\alpha\beta), & \cos(\alpha\gamma), & \cos(\alpha\delta), & \alpha\div r', \\ \cos(\beta\alpha), & -1, & \cos(\beta\gamma), & \cos(\beta\delta), & \beta\div r'', \\ \cos(\gamma\alpha), & \cos(\gamma\beta), & -1, & \cos(\gamma\delta), & \gamma\div r''', \\ \cos(\delta\alpha), & \cos(\delta\beta), & \cos(\delta\gamma), & -1 & \delta\div r'''', \\ \alpha\div r' & \beta\div r'' & \gamma\div r''' & \delta\div r'''' & 0. \end{vmatrix} \quad . \quad . \quad (29)$$

Cor. Hence, if the four spheres of reference $\alpha, \beta, \gamma, \delta$ be mutually orthogonal, the equation becomes

$$-U^2 \equiv \left(\frac{\alpha}{r'}\right)^2 + \left(\frac{\beta}{r''}\right)^2 + \left(\frac{\gamma}{r'''}\right)^2 + \left(\frac{\delta}{r}\right)^2 = 0;$$

and by incorporating constants with the variables it becomes $\alpha^2+\beta^2+\gamma^2+\delta^2=0$.

We shall find the value of U^2 in this latter form very important.

29. If the tetrahedron to which F is referred be inscribed in F, the coefficients a, b, c, d vanish; then forming the tangential equation and replacing $\lambda, \mu, \nu, \varrho$ by $\alpha, \beta, \gamma, \delta$, we have the following theorem. If the equation of a cyclide be in the form

$$\begin{vmatrix} 0, & n, & m, & p, & \alpha, \\ n, & 0, & l, & q, & \beta, \\ m, & l, & 0, & r, & \gamma, \\ p, & q, & r, & 0, & \delta, \\ \alpha, & \beta, & \gamma, & \delta, & 0 \end{vmatrix} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

that is, of a symmetrical determinant bordered with the variables whose diagonal terms are each zero, the spheres of reference have each double contact with the cyclide; in other words, they are generating spheres.

Cor. From this theorem, combined with article 3, we can easily get the equation of a sphere circumscribing a tetrahedron.

30. If the equation of a cyclide be given in the form

$$\left.\begin{aligned} \alpha^2+\beta^2+\gamma^2+\delta^2+2l(\beta\gamma+\alpha\delta) \\ +2m(\alpha\gamma+\beta\delta)+2n(\alpha\beta+\gamma\delta)=0 \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

where $1+2\,lmn=l^2+m^2+n^2$, it can be proved, precisely as in SALMON's 'Geometry of Three Dimensions,' p. 153, that each of the spheres of reference cuts the cyclide in two circles. Hence (see art. 19, *Cor.* 1) each of the spheres of references is a generating sphere.

31. If the coefficients of $\alpha^2, \beta^2, \gamma^2, \delta^2$ in the general equation of a cyclide vanish, then the coefficients of $\lambda^2, \mu^2, \nu^2, \varrho^2$ vanish in the tangential equation of the focal quadric; and hence *if the coefficients of the squares of the variables vanish in the equation of a*

cyclide, the focal quadric is inscribed in the tetrahedron formed by the centres of the spheres of reference.

32. Let $W \equiv (a, b, c, d, l, m, n, p, q, r \between \alpha, \beta, \gamma, \delta)^2 = 0$ be the equation of a cyclide, and we know that the square of the Jacobian of α, β, γ, δ is given by the equation (29). Now, substituting λ, μ, ν, ϱ for α, β, γ, δ in these equations, we have the tangential equations of the focal quadric F of the cyclide and the sphere U; but if F and U be referred to their common self-conjugate tetrahedron, their equations will take the form

$$a\lambda^2 + b\mu^2 + c\nu^2 + d\varrho^2 = 0, \quad \lambda^2 + \mu^2 + \nu^2 + \varrho^2 = 0.$$

Hence we have W and U^2 given by the equations

$$W \equiv a\alpha^2 + b\beta^2 + c\gamma^2 + d\delta^2 = 0,$$
$$U^2 \equiv -(\alpha^2 + \beta^2 + \gamma^2 + \delta^2) = 0 \text{ (see art. 28. } Cor.).$$

Now if, for the sake of uniformity, we represent U by ε, we have the following equation identically true,

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \varepsilon^2 = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

and since the addition of any multiple of an expression which vanishes identically does not alter a function, we see that the equation of a cyclide may be written in the following form by adding $e(\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \varepsilon^2)$ to $a\alpha^2 + b\beta^2 + c\gamma^2 + d\delta^2 = 0$, and afterwards putting a for $(a+e)$, b for $(b+e)$ &c.,

$$W = a\alpha^2 + b^2\beta + c\gamma^2 + d\delta^2 + e\varepsilon^2 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

in which each of the spheres of reference is cut orthogonally by all the others.

We could show *à priori* that W can be expressed in either of the forms

$$a\alpha^2 + b\beta^2 + c\gamma^2 + d\delta^2 = 0,$$

or

$$a\alpha^2 + b\beta^2 + c\gamma^2 + d\delta^2 + e\varepsilon^2 = 0;$$

for the first form contains explicitly three constants, and each sphere contains four constants, so that there are $3 + 4 \times 4 = 19$ constants at our disposal; but each pair of spheres being mutually orthogonal is equivalent to six conditions. Hence we have thirteen constants, which is the number required. Similarly, in the second form we have twenty-four constants; but these are subject to the equation of condition $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \varepsilon^2$, which does not vanish identically except by the incorporation of certain constants, and the condition of orthogonality of each pair of spheres is equivalent to ten conditions. Hence, as before, we have thirteen conditions remaining.

33. By means of the identical relation $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \varepsilon^2 = 0$, we can eliminate in succession each of the five spheres α, β, γ, δ, ε from the equation (33) of the cyclide; and we see that the same cyclide can be written in five different ways, the letter eliminated representing the sphere used in the corresponding equation of the surface. Hence we have

$$\left.\begin{array}{ll}
W=(a-b)\beta^2+(a-c)\gamma^2+(a-d)\delta^2+(a-e)\varepsilon^2=0, & (\text{I}) \\
W=(b-c)\gamma^2+(b-d)\delta^2+(b-e)\varepsilon^2+(b-a)\alpha^2=0, & (\text{II}) \\
W=(c-d)\delta^2+(c-e)\varepsilon^2+(c-a)\alpha^2+(c-b)\beta^2=0, & (\text{III}) \\
W=(d-e)\varepsilon^2+(d-a)\alpha^2+(d-b)\beta^2+(d-c)\gamma^2=0, & (\text{IV}) \\
W=(e-a)\alpha^2+(e-b)\beta^2+(e-c)\gamma^2+(e-d)\delta^2=0. & (\text{V})
\end{array}\right\} \quad \ldots \quad (34)$$

If we denote the focal quadrics corresponding to these different forms of the equation of W by F F′ F″ F‴ $F^{(iv)}$, we get the following as the tangential equations of the five focal quadrics:—

$$\left.\begin{array}{l}
F\ \ =(a-b)\mu^2+(a-c)\nu^2+(a-d)\varrho^2+(a-e)\sigma^2=0, \\
F'\ =(b-c)\nu^2+(b-d)\varrho^2+(b-e)\sigma^2+(b-a)\lambda^2=0, \\
F''=(c-d)\varrho^2+(c-e)\sigma^2+(c-a)\lambda^2+(c-b)\mu^2=0, \\
F'''=(d-e)\sigma^2+(d-a)\lambda^2+(d-b)\mu^2+(d-c)\nu^2=0, \\
F^{(iv)}=(e-a)\lambda^2+(e-b)\mu^2+(e-c)\nu^2+(e-d)\varrho^2=0.
\end{array}\right\} \quad \ldots \quad (35)$$

So that the cyclide W can be generated in five different ways as the envelope of a variable sphere whose centre moves on a quadric and cuts a given sphere orthogonally; the corresponding spheres and quadrics for generating W being α, F; β, F′; γ, F″; δ, F‴; ε, $F^{(iv)}$ respectively.

34. Since the tangential equation of α is plainly $\mu^2+\nu^2+\varrho^2+\sigma^2=0$,

of β, $\nu^2+\varrho^2+\sigma^2+\lambda^2$, &c.,

we get the equations of the developables circumscribed to the pairs of quadrics α, F; β, F′, &c. by a known process; thus the developable circumscribed about α and F will be the envelope of the quadric, whose tangential equation

$$(a-b)\mu^2+(a-c)\nu^2+(a-d)\varrho^2+(a-e)\sigma^2+k(\mu^2+\nu^2+\varrho^2+\sigma^2)=0;$$

and by taking $k=(b-a)$, $(c-a)$, $(d-a)$, $(e-a)$ in succession, we see that the double lines of the developable are the plane conics, whose tangential equations are:

$$\left.\begin{array}{l}
(b-c)\nu^2+(b-d)\varrho^2+(b-e)\sigma^2=0, \\
(c-b)\mu^2+(c-d)\varrho^2+(c-e)\sigma^2=0, \\
(d-b)\mu^2+(d-c)\nu^2+(d-e)\sigma^2=0, \\
(e-b)\mu^2+(e-c)\nu^2+(e-d)\varrho^2=0.
\end{array}\right\} \quad \ldots \quad (36)$$

By comparing these with the system of equations (35), we see that the first conic is a plane section of the quadric F′, the second of F″, the third of F‴, and the fourth of F⁗.

Hence, if we call the spheres α, β, γ, δ, ε the spheres of inversion of the cyclide (we shall prove this in a future Chapter), and call Σ, Σ', Σ'', Σ''', Σ'''' the five developables circumscribed to the spheres of inversion and their corresponding focal quadrics, we have the following theorem:—

The double lines of Σ	*are plane*	*sections of*	F', F'', F''', F'''',
,,	Σ'	,,	F'', F''', F'''', F,
,,	Σ''	,,	F''', F'''', F, F',
,,	Σ'''	,,	F'''', F, F', F'',
,,	Σ''''	,,	F, F', F'', F'''.

35. If we take the first of the equations (34) to represent W, the corresponding sphere of inversion is α^2; but this, in virtue of the identical relation $\alpha^2+\beta^2+\gamma^2+\delta^2+\varepsilon^2=0$, is also given by the equation $\beta^2+\gamma^2+\delta^2+\varepsilon^2=0$; and eliminating β^2, γ^2, δ^2, ε^2 in succession between W and α^2, we see that each of the four binodal cyclides are inscribed in W,

$$\left.\begin{aligned}(b-c)\gamma^2+(b-d)\delta^2+(b-e)\varepsilon^2=0,\\(c-b)\beta^2+(c-d)\delta^2+(c-e)\varepsilon^2=0,\\(d-b)\beta^2+(d-c)\gamma^2+(d-e)\varepsilon^2=0,\\(e-b)\beta^2+(e-c)\gamma^2+(e-d)\delta^2=0,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

and these cyclides have the double lines of Σ as focal quadrics.

It is plain that we get corresponding results for each of the five forms (34) in which the equation of W may be written, so that *the cyclide* W *is circumscribed about ten binodal cyclides. The focal quadrics of these binodals are plane conics, and through each conic two developables pass.*

SECTION II.—*Sphero-quartics.*

36. Let P, a point on the surface of the sphere U, be the centre of the small circle S on the surface of the same sphere, O a fixed point, also on the surface of U, which we shall take as origin, OX a great circle of U corresponding to the initial line in plane geometry, and let $OP=n$ and the angle $POX=m$; then m and n are what I shall call the spherical coordinates of the point P; and whenever I shall use the term spherical coordinates it is in the sense here explained. Now let θ and ρ be the spherical coordinates of any point Q of the circle S, then (the reader can easily construct the figure) we have from the spherical triangle OPQ, r being the radius of S,

$$\cos r=\cos n\cos\rho+\sin n\sin\rho\cos(\theta-m). \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

This equation may be taken as the equation of the small circle S. Now if in the equation (38) we substitute the spherical coordinates of any point Q′ whose distance from P is the arc r', we plainly get $\cos r-\cos r'$; but $\cos r-\cos r'$ is equal to the perpendicular let fall from the point Q′ on the plane of the small circle S, hence we have the following theorem:—*The result of substituting the spherical coordinates of any point on the surface of a sphere radius unity in the equation of any small circle on the sphere is equal to the perpendicular distance of the point from the plane of the small circle.*

37. If any sphere Ω intersect a cyclide W, the curve of intersection is a sphero-quartic.

Demonstration. Let $W=a\alpha^2+b\beta^2+c\gamma^2+d\delta^2$, and let perpendiculars from any point P

of the curve $W\Omega$ on the planes through the intersection of Ω and α, Ω and β, &c. be denoted by x, y, z, w; then if the distances from the centre of Ω to the centres of $\alpha, \beta, \gamma, \delta$ be denoted by $\omega_1, \omega_2, \omega_3, \omega_4$ respectively, it is easy to see that the results of substituting the coordinates of P in $\alpha, \beta, \gamma, \delta$ are $2\omega_1x$, $2\omega_2y$, $2\omega_3z$, $2\omega_4w$, and therefore the quadric

$$a\omega_1^2x^2+b\omega_2^2y^2+c\omega_3^2z^2+d\omega_4^2w^2=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

passes through the curve $W\Omega$. Hence the curve $W\Omega$ is also the intersection of the sphere Ω and the quadric (39), and therefore it is a sphero-quartic.

38. If in the last article we suppose the sphere Ω to coincide with U, the sphere orthogonal to $\alpha, \beta, \gamma, \delta$, and if we denote the circles in which the spheres $\alpha, \beta, \gamma, \delta$ intersect U_1 by the same notation, that is, by $\alpha, \beta, \gamma, \delta$, then if the radii of the circles $\alpha, \beta, \gamma, \delta$ be r', r'', r''', r'''', it is plain that $\omega_1, \omega_2, \omega_3, \omega_4$ of the last article become $\sec r'$, $\sec r''$, $\sec r'''$, $\sec r''''$, the radius of U being denoted by unity. Hence, by articles 36 and 37, we have the following theorem:—*If* $W=(a, b, c, d, l, m, n, p, q, r\between\alpha, \beta, \gamma, \delta)^2=0$ *be the equation of a cyclide, the equation of the sphero-quartic* WU *will be*

$$(a, b, c, d, l, m, n, p, q, r)\left(\frac{\alpha}{\cos r'}, \frac{\beta}{\cos r''}, \frac{\gamma}{\cos r'''}, \frac{\delta}{\cos r''''}\right)^2=0, \quad . \quad . \quad . \quad (40)$$

where $\alpha, \beta, \gamma, \delta$ *are the small circles of intersection of the spheres* $\alpha, \beta, \gamma, \delta$ *with* U.

39. From the three last articles, combined with art. 28, we have at once the following theorem, which is a very important one in the theory of sphero-quartics:—*If* $\alpha, \beta, \gamma, \delta$ *be any four small circles on the sphere* U, *the following relation will be true for any point on the surface of* U, *and will therefore be an identical relation*:

$$\begin{vmatrix} -1, & \cos(\alpha\beta), & \cos(\alpha\gamma), & \cos(\alpha\delta), & \alpha\div\sin r', \\ \cos(\beta\alpha), & -1, & \cos(\beta\gamma), & \cos(\beta\delta), & \beta\div\sin r'', \\ \cos(\gamma\alpha), & \cos(\gamma\beta), & -1, & \cos(\gamma\delta), & \gamma\div\sin r''', \\ \cos(\delta\alpha), & \cos(\delta\beta), & \cos(\delta\gamma), & -1, & \delta\div\sin r'''', \\ \alpha\div\sin r', & \beta\div\sin r'', & \gamma\div\sin r''', & \delta\div\sin r'''', & 0, \end{vmatrix}=0. \quad . \quad . \quad (41)$$

Cor. If the circles $\alpha, \beta, \gamma, \delta$ on the surface of U be mutually orthogonal, the relation is identically true for any point on U,

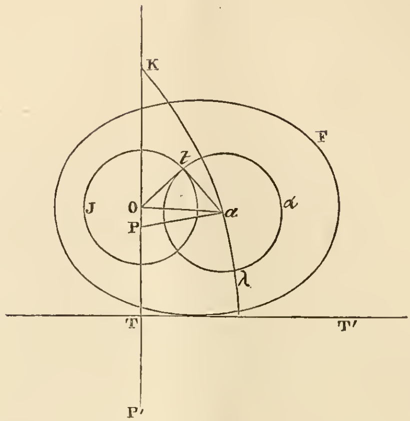
$$\frac{\alpha^2}{\sin^2 r'}+\frac{\beta^2}{\sin^2 r''}+\frac{\gamma^2}{\sin^2 r'''}+\frac{\delta^2}{\sin^2 r''''}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

If we incorporate the constants with the variables this becomes $\alpha^2+\beta^2+\gamma^2+\delta^2=0$.

40. If α, β, γ be three small circles on the sphere U, and if a sphero-quartic be given by an equation of the second degree $(a, b, c, f, g, h\between\alpha, \beta, \gamma)^2=0$, I say the tangential equation of the corresponding focal sphero-conic is

$$(a, b, c, f, g, h\between\lambda, \mu, \nu)^2=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

Fig. 1.



Demonstration. Let F be the focal sphero-conic, a the centre of α, one of the circles of reference; then if P, P′ be limiting points of the system, composed of J and any tangent TT′ to F; P, P′ are points on the sphero-quartic. Let K be the pole of the great circle TT′. Now if at be the tangent from a to J, by art. 36, the result of substituting the coordinates of P in the small circle $\alpha = \cos at - \cos a\text{P}$, this may be written

$$\alpha = \cos at - \cos a\text{P};$$

but

$$\cos at = \frac{\cos a\text{O}}{\cos \text{O}t} = \frac{\sin\lambda \sin \text{OT} + \cos\lambda \cos \text{OT} \cos \text{OK}a}{\cos \text{O}t},$$

and

$$\cos a\text{P} = \sin\lambda \sin \text{PT} + \cos\lambda \cos \text{PT} \cos \text{OK}a$$

$$= \sin\lambda \sin \text{PT} + \frac{\cos\lambda \cos \text{OT} \cos \text{OK}a}{\cos \text{O}t}.$$

Hence, by substitution, we get

$$\alpha = \sin\lambda \left\{ \frac{\sin \text{OT}}{\cos \text{O}t} - \sin \text{PT} \right\};$$

and putting for $\cos \text{O}t$ its value $\cos \text{OT} \div \cos \text{PT}$ (see art. 24), we get

$$\alpha = \frac{\sin\lambda \sin \text{OP}}{\cos \text{OT}}.$$

Hence the results of substituting the coordinates of any point P of the sphero-quartic in the equation of the small circles α, β, γ are proportional to the sines of the arcs from the centres of α, β, γ to a great circle tangential to the sphero-conic F, and hence the proposition is proved.

41. If the cyclide W be expressed in terms of four spheres α, β, γ, δ which are mutually orthogonal, then the sphero-quartic WU will be expressed in terms of four circles which

are mutually orthogonal, and its equation will be of the form $a\alpha^2+b\beta^2+c\gamma^2+d\delta^2=0$; but $\alpha^2+\beta^2+\gamma^2+\delta^2=0$ is an identical relation. Hence, eliminating each of the variables α^2, β^2, γ^2, δ^2 in succession, we see that the same sphero-quartic may be expressed by either of the four equations:

$$\left.\begin{aligned}(a-b)\beta^2+(a-c)\gamma^2+(a-d)\delta^2=0,\\(b-a)\alpha^2+(b-c)\gamma^2+(b-d)\delta^2=0,\\(c-a)\alpha^2+(c-b)\beta^2+(c-d)\delta^2=0,\\(d-a)\alpha^2+(d-b)\beta^2+(d-c)\gamma^2=0,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

and by the last article we see that the sphero-quartic has four focal sphero-conics, whose tangential equations are:

$$\left.\begin{aligned}(a-b)\mu^2+(a-c)\nu^2+(a-d)\varrho^2=0,\\(b-a)\lambda^2+(b-c)\nu^2+(b-d)\varrho^2=0,\\(c-a)\lambda^2+(c-b)\mu^2+(c-d)\varrho^2=0,\\(d-a)\lambda^2+(d-b)\mu^2+(d-c)\nu^2=0.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

Cor. *Sphero-quartics may be generated in four different ways as the envelope of a variable circle* which cuts a given circle orthogonally, and whose *centre moves along a given sphero-conic.*

42. If $W=(a, b, c, d, l, m, n, p, q, r\between\alpha, \beta, \gamma, \delta)^2=0$ be the general equation of a cyclide, and U the sphere orthogonal to α, β, γ, δ, then it is easy to see that the results of substituting the coordinates of any point P of the sphero-quartic (WU) in the equations of α, β, γ, δ are proportional to the perpendiculars from the centres of α, β, γ, δ on the tangent plane to U at the point P; but if these be perpendiculars to λ, μ, ν, ϱ, we see that the surface whose tangential equation is

$$(a, b, c, d, l, m, n, p, q, r\between\lambda, \mu, \nu, \varrho)^2$$

is inscribed in the developable formed by the tangent planes to U along the sphero-quartic WU, but this tangential equation is that of the focal quadric of W. Hence we have the following theorem:—*The developable circumscribed about the focal quadric of a cyclide and the corresponding sphere of inversion* U *touches the sphere along the sphero-quartic* (WU), *and the cones whose vertices are at the centre of* U, *and which stand on the nodal conics of the developable, intersect* U *in the focal sphero-conics of the sphero-quartic* WU. The latter part of the theorem is evident by writing the equation of the cyclide in terms of four spheres mutually orthogonal, and from the equations (45) of the last article*.

* [We have given in art. 33 the equations in tangential coordinates of the five focal quadrics of a cyclide; the following investigation gives, being given the equations of a focal quadric and the corresponding sphere of inversion in Cartesian coordinates, the equations in Cartesian coordinates of the four remaining focal quadrics.

I. Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$ be the focal quadric F of a cyclide W, and $(x-f)^2+(y-g)^2+(z-h)^2-r^2=0$ be the corresponding sphere of inversion; then if from the centre O of the sphere we let fall a perpendicular OT

CHAPTER IV.

Sphero-quartics (continued).

43. In discussing the properties of sphero-quartics, we have hitherto considered a sphero-quartic as the intersection of a sphere and a cyclide. There is another mode of considering sphero-quartics, which offers many advantages for the investigation of these curves, namely, to consider a sphero-quartic as the curve of intersection of a sphere and a quartic cone tangential to the cyclide, the vertex of the cone being at the centre of the sphere, which we shall take as one of the spheres of inversion of the cyclide. This method of studying the sphero-quartic will give us an opportunity of showing the connexion which exists between the invariants and covariants of plane conics and of circles

---

on any tangent plane to F, and take two points P, P′ in opposite directions from T on OT so that

$$\mathrm{OT}^2-\mathrm{TP}^2=\mathrm{OT}^2-\mathrm{TP}'^2=r^2,$$

the locus of the points P, P′ is the cyclide W; but denoting OT by p, and OP by ρ, this gives us $2p\rho=r^2+\rho^2$, or

$$2\sqrt{a^2\cos^2\alpha+b^2\cos^2\beta+c^2\cos^2\gamma}-2(f\cos\alpha+g\cos\beta+h\cos\gamma)=r^2+\rho^2,$$

$\cos\alpha$, $\cos\beta$, $\cos\gamma$ being the direction cosines of OT. Hence, if the centre of the sphere be now taken as origin, we have the equation of the cyclide

$$4(a^2x^2+b^2y^2+c^2z^2)=(x^2+y^2+z^2+2fx+2gy+2hz+r^2)^2.$$

II. The equation of the cyclide given in I. is the envelope of the quadric $\mathrm{S}+\mu\mathrm{C}+\mu^2=0$, where S represents the cone $a^2x^2+b^2y^2+c^2z^2$, and C the sphere $x^2+y^2+z^2+2fx+2gy+2hz+r^2$; and the condition that this should represent a cone is the discriminant

$$(a^2+\mu)(b^2+\mu)(c^2+\mu)(\mu r^2+\mu^2)-\mu^2f^2(b^2+\mu)(c^2+\mu)-\mu^2g^2(c^2+\mu)(a^2+\mu)-\mu^2h^2(a^2+\mu)(b^2+\mu)=0,$$

or, as it may be written,

$$\frac{\mu^2f^2}{a^2+\mu}+\frac{\mu^2g^2}{b^2+\mu}+\frac{\mu^2h^2}{c^2+\mu}=\mu r^2+\mu^2.$$

If the five values of μ in this equation be denoted by μ_1, μ_2, μ_3, μ_4, μ_5, we have the equations of the five cones which have double contact with the cyclide (see art. 187), $\mathrm{S}+\mu_1\mathrm{C}+\mu_1^2$, $\mathrm{S}+\mu_2\mathrm{C}+\mu_2^2$, &c.; and the vertices of these five cones are, by the same article, the five centres of inversion of the cyclide. Since one value is obviously $=0$ in the foregoing equation, we see that the cone whose vertex is the centre of the sphere of inversion

$$(x-f)^2+(y-g)^2+(z-h)^2-r^2=0$$

will be, when that centre is taken as origin,

$$a^2x^2+b^2y^2+c^2z^2=0.$$

Hence, if the other centres be taken respectively as origin, the equations of the other cones will be

$$(a^2+\mu_2)x^2+(b^2+\mu_2)y^2+(c^2+\mu_2)z^2=0 \quad . \quad . \quad . \quad (\alpha')$$

$$(a^2+\mu_3)x^2+(b^2+\mu_3)y^2+(c^2+\mu_3)z^2=0 \quad . \quad . \quad . \quad (\beta')$$

$$(a^2+\mu_4)x^2+(b^2+\mu_4)y^2+(c^2+\mu_4)z^2=0 \quad . \quad . \quad . \quad (\gamma')$$

$$(a^2+\mu_5)x^2+(b^2+\mu_5)y^2+(c^2+\mu_5)z^2=0 \quad . \quad . \quad . \quad (\delta')$$

Now, since the cone $a^2x^2+b^2y^2+c^2z^2=0$ is the reciprocal of the asymptotic cone of the focal quadric $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$, we infer that the cones (α'), (β'), (γ'), (δ') are the reciprocals of the asymptotic cones of the

on the sphere, and to show that sphero-quartics may be generated in the same way as I have given in the Fifth Chapter of my 'Bicirculars' for generating plane quartics having two finite double points.

44. The equation of a right cone whose semivertical angle is ϱ is

$$L^2-\cos^2\varrho(x^2+y^2+z^2)=0, \qquad (46)$$

where L is a plane through the vertex of the cone: now this cone intersects a sphere of radius unity whose centre is at its vertex in two small circles; and I say that the two factors of the equation (46), namely $L\pm\cos\varrho(x^2+y^2+z^2)^{\frac{1}{2}}$, may be taken to represent these two circles; for the equation (38), which represents a small circle on the sphere, will become by transformation to three rectangular planes $L=\cos\varrho(x^2+y^2+z^2)^{\frac{1}{2}}$, and its twin circle will be the other factor, $L+\cos\varrho(x^2+y^2+z^2)^{\frac{1}{2}}$.

---

other four focal quadrics; and hence we have the following system of Cartesian equations of these focal quadrics:—

$$\frac{x^2}{a^2+\mu_2}+\frac{y^2}{b^2+\mu_2}+\frac{z^2}{c^2+\mu_2}=1,$$

$$\frac{x^2}{a^2+\mu_3}+\frac{y^2}{b^2+\mu_3}+\frac{z^2}{c^2+\mu_3}=1,$$

$$\frac{x^2}{a^2+\mu_4}+\frac{y^2}{b^2+\mu_4}+\frac{z^2}{c^2+\mu_4}=1,$$

$$\frac{x^2}{a^2+\mu_5}+\frac{y^2}{b^2+\mu_5}+\frac{z^2}{c^2+\mu_5}=1,$$

so that the five focal quadrics are confocal, as we know otherwise.

III. Since the equation

$$\frac{\mu^2f^2}{a^2+\mu}+\frac{\mu^2g^2}{b^2+\mu}+\frac{\mu^2h^2}{c^2+\mu}=\mu^2+\mu r^2$$

may be written in the form

$$\frac{f^2}{a^2+\mu}+\frac{g^2}{b^2+\mu}+\frac{h^2}{c^2+\mu}=1+\frac{r^2}{\mu},$$

and this is the discriminant of $\mu F+J$, where

$$F\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0, \quad J\equiv(x-f)^2+(y-g)^2+(z-h)^2-r^2=0$$

(see SALMON's 'Geometry of Three Dimensions,' p. 146), we infer that the same values which will make $\mu F+J$ a cone will also make $S+\mu C+\mu^2$ (see II.) a cone. The two cones will have a common vertex, their equations referred to that vertex as origin being

$$\frac{x^2(a^2+\mu)}{a^2}+\frac{y^2(b^2+\mu)}{b^2}+\frac{z^2(c^2+\mu)}{c^2}=0,$$

$$x^2(a^2+\mu)+y^2(b^2+\mu)+z^2(c^2+\mu)=0.$$

Hence we have the following remarkable theorem:—If F and J be a corresponding focal quadric and sphere of inversion of a cyclide, and if μ_1, μ_2, μ_3, μ_4 be the four roots of the biquadratic which is the discriminant of $\mu F+J$, then if F be given in its canonical form,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0,$$

the equations of the four other focal quadrics are got from this by changing a^2, b^2, c^2 respectively into $(a^2+\mu_1)$, $(b^2+\mu_1)$, $(c^2+\mu_1)$, &c.

Now if we put S for the point sphere $x^2+y^2+z^2$, the equations of the two circles may be written

$$S^{\frac{1}{2}}\pm L\sec\varrho=0. \qquad (47)$$

It is clear that these equations (47) may also be interpreted as denoting separately the two sheets of the cone (46)—that is $S^{\frac{1}{2}}+L\sec\varrho$ represents one sheet of it, and $S^{\frac{1}{2}}-L\sec\varrho$ the other. Hence we infer from this article and from article 38 *that the equation*

$$(a, b, c, d, l, m, n, p, q, r\between\alpha, \beta, \gamma, \delta)^2=0$$

will represent a cyclide, a sphero-quartic, or a tangent cone to the cyclide, whose vertex is

---

IV. When $S+\mu C+\mu^2$ (see II.) represents a cone, the coordinates of the vertex are, by the usual process,

$$\frac{-\mu f}{a^2+\mu},\ \frac{-\mu g}{b^2+\mu},\ \frac{-\mu h}{c^2+\mu},$$

if referred to the centre of J as origin, or

$$\frac{a^2 f}{a^2+\mu},\ \frac{b^2 g}{b^2+\mu},\ \frac{c^2 h}{c^2+\mu},$$

if referred to the centre of F as origin. Hence we have the following theorem:—If $F\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$, and $J\equiv(x-f)^2+(y-g)^2+(z-h)^2-r^2=0$ be a corresponding focal quadric and sphere of inversion of a cyclide, and if μ_1, μ_2, μ_3, μ_4 be the four roots of the biquadratic in μ, which is the discriminant of $\mu F+J$, then the coordinates of the centres of the other four spheres of inversion are:

$$\frac{a^2 f}{a^2+\mu_1},\ \frac{b^2 g}{b^2+\mu_1},\ \frac{c^2 h}{c^2+\mu_1},$$
$$\frac{a^2 f}{a^2+\mu_2},\ \frac{b^2 g}{b^2+\mu_2},\ \frac{c^2 h}{c^2+\mu_2},$$
$$\frac{a^2 f}{a^2+\mu_3},\ \frac{b^2 g}{b^2+\mu_3},\ \frac{c^2 h}{c^2+\mu_3},$$
$$\frac{a^2 f}{a^2+\mu_4},\ \frac{b^2 g}{b^2+\mu_4},\ \frac{c^2 h}{c^2+\mu_4}.$$

Cor. These values satisfy the system of determinants

$$\left\|\begin{matrix} \frac{x}{a^2}, & \frac{y}{b^2}, & \frac{z}{c^2}, \\ (x-f), & (y-g), & (z-h), \end{matrix}\right\|=0,$$

or

$$\left|\begin{matrix} \partial_x & \partial_y & \partial_z \\ \hline , & , & , \\ , & , & , \end{matrix}\ \middle|\ \begin{matrix} \\ F, \\ J, \end{matrix}\right.=0.$$

Hence we have the following theorem:—If F and J be a corresponding focal quadric and sphere of inversion of a cyclide, then the five centres of inversion of the cyclide lie on the Jacobian curve of J and F (see CAYLEY, "Memoir on Quartic Surfaces," Proceedings of the London Mathematical Society).

V. Being given

$$F\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0,\quad J\equiv(x-f)^2+(y-g)^2+(z-h)^2-r^2=0,$$

the equation of the cyclide is

$$4(a^2x^2+b^2y^2+c^2z^2)-(x^2+y^2+z^2+2fx+2gy+2hz+r^2)^2=0. \qquad (A)$$

at the centre of the sphere of inversion U *of the cyclide, according as we regard the variables* α, β, γ, δ *as spheres, as circles on the sphere* U, *the Jacobian of the spheres* α, β, γ, δ, *or as single sheets of cones having their common vertex at the centre of* U.

45. From the double interpretation of the equation $S^{\frac{1}{2}}+L\sec\rho=0$ as denoting a small circle on the sphere, or as denoting a single sheet of the cone (46), all the results which we shall prove in the following articles are twofold in their application; for simplicity, however, I shall consider it as denoting a circle unless the contrary is expressed. If the equation of the plane L be $ax+by+cz=0$, it is clear that a, b, c may be regarded either as the direction cosines of L, or the coordinates of its pole on the sphere U, for

---

Again, being given

$$F' \equiv \frac{x^2}{a^2+\mu_1}+\frac{y^2}{b^2+\mu_1}+\frac{z^2}{c^2+\mu_1}-1=0,$$

$$J' \equiv \left(x-\frac{a^2f}{a^2+\mu_1}\right)^2+\left(y-\frac{b^2g}{b^2+\mu_1}\right)^2+\left(z-\frac{c^2h}{c^2+\mu_1}\right)^2-r'^2=0,$$

the equation of the cyclide is

$$\left.\begin{aligned}&4\{(a^2+\mu_1)x^2+(b^2+\mu_1)y^2+(c^2+\mu_1)z^2\}\\ &\quad=x^2+y^2+z^2+\frac{2a^2f}{a^2+\mu_1}x+\frac{2b^2g}{b^2+\mu_1}y+\frac{2c^2h}{c^2+\mu_1}z+r'^2.\end{aligned}\right\} \quad \text{(B)}$$

The origin in equation (A) is the centre of J, and the origin in equation (B) is the centre of J′, that is, the point whose coordinates with respect to the centre of J are

$$\frac{-\mu_1 f}{a^2+\mu_1},\ \frac{-\mu_1 g}{b^2+\mu_1},\ \frac{-\mu_1 h}{c^2+\mu_1}.$$

In order to compare the equations (A) and (B), which represent the same surface, we must transform (B) to the same origin as (A), or (A) to the same origin as (B): we will adopt the latter transformation, and we get the following result:—

$$\left.\begin{aligned}&4\left\{a^2\left(x+\frac{\mu_1 f}{a^2+\mu_1}\right)^2+b^2\left(y+\frac{\mu_1 g}{b^2+\mu_1}\right)^2+c^2\left(z+\frac{\mu_1 h}{c^2+\mu_1}\right)^2\right\}\\ &=\left\{\left(x+\frac{\mu_1 f}{a^2+\mu_1}\right)^2+\left(y+\frac{\mu_1 g}{b^2+\mu_1}\right)^2+\left(z+\frac{\mu_1 h}{c^2+\mu_1}\right)^2\right.\\ &\quad\left.+2f\left(x+\frac{\mu_1 f}{a^2+\mu_1}\right)+2g\left(y+\frac{\mu_1 g}{b^2+\mu_1}\right)+2h\left(z+\frac{\mu_1 h}{c^2+\mu_1}\right)+r^2\right\}^2.\end{aligned}\right\} \quad \text{(C)}$$

Since the equations (B) and (C) represent the same cyclide and are referred to the same origin, by comparing the absolute terms, we shall get the value of r^2 in terms of r'^2, μ, and known constants. The absolute term in equation (C) is

$$\left\{\left(\frac{\mu_1 f}{a^2+\mu_1}+f\right)^2+\left(\frac{\mu_1 g}{b^2+\mu_1}+g\right)^2+\left(\frac{\mu_1 h}{c^2+\mu_1}+h\right)^2-(f^2+g^2+h^2)+r^2\right\}^2$$
$$-4\left\{\frac{a^2\mu_1^2 f^2}{(a^2\mu_1)^2}+\frac{b^2\mu_1^2 g^2}{(b^2+\mu_1)^2}+\frac{c^2\mu_1^2 h^2}{(c^2+\mu_1)^2}\right\},$$

which, being reduced by means of the relation

$$\frac{f^2}{a^2+\mu_1}+\frac{g^2}{b^2+\mu_1}+\frac{h^2}{c^2+\mu_1}=1+\frac{r^2}{\mu_1},$$

becomes

$$\left\{\frac{\mu_1^2 f^2}{(a^2+\mu_1)^2}+\frac{\mu_1^2 g^2}{(b^2+\mu_1)^2}+\frac{\mu_1^2 h^2}{(c^2+\mu_1)^2}-r^2\right\}^2:$$

these are both the same thing. Hence it follows, when we represent a circle on U by the equation $S^{\frac{1}{2}}-L=0$, where $L\equiv ax+by+cz=0$, that a, b, c are equal to the direction cosines of L multiplied respectively by the secant of the spherical radius of the circle.

46. Now let us take, as in conics, the small circles

$$S^{\frac{1}{2}}-L=0,\quad S^{\frac{1}{2}}=M=0,$$

and form the invariants of this system; thus

$$S^{\frac{1}{2}}-L+k(S^{\frac{1}{2}}-M)=0$$

is a small circle coaxal with $S^{\frac{1}{2}}-L$ and $S^{\frac{1}{2}}-M$, $L-M=0$ being the great circle through

---

and the absolute term in equation (B) is r'^4. Hence we get

$$r^2+r'^2=\frac{\mu_1^2 f^2}{(a^2+\mu_1)^2}+\frac{(\mu_1^2 g^2}{(b^2+\mu_1)^2}+\frac{\mu_1^2 h^2}{(c^2+\mu_1)^2}.$$

That is the sum of the squares of the radii of J and J$'$= square of the distance between their centres, and hence J and J$'$ cut orthogonally.

VI. The cyclide got from J and F is the envelope of the quadric $S+\mu C+\mu^2$, where

$$S\equiv a^2x^2+b^2y^2+c^2z^2,\quad C\equiv x^2+y^2+z^2+2fx+2gy+2hz+r^2.$$

The same cyclide, got from J$'$ and F$'$, is the envelope of the quadric $S'+\lambda C'+\lambda^2$, where

$$S'\equiv(a^2+\mu_1)x^2+(b^2+\mu_1)y^2+(c^2+\mu_1)z^2,$$

and

$$C'\equiv(x^2+y^2+z^2)+\frac{2a^2f}{a^2+\mu_1}x+\frac{2b^2g}{b^2+\mu_1}y+\frac{2c^2h}{c^2+\mu_1}z+r'^2.$$

Now, to show that $S+\mu C+\mu^2$ and $S'+\lambda C'+\lambda^2$ represent different quadrics, we are to observe that the first is referred to the centre of J as origin, and the second to the centre of J$'$ as origin. Now let us transform the first to the same origin as the second; we must change x into $x-\frac{\mu_1 f}{a^2+\mu_1}$, y into $y-\frac{\mu_1 g}{b^2+\mu_1}$, z in to $z-\frac{\mu_1 h}{c^2+\mu_1}$; and, in order that they may be identical, we must have $\mu=\mu_1+\lambda$; this will make the coefficients of x^2, y^2, z^2 the same in both, but the coefficients of x, y, z will be different. Hence $S+\mu C+\mu^2$ and $S'+\lambda C'+\lambda^2$ cannot represent the same quadric. Hence we have the following theorem:—*A cyclide which has no node may be generated in five different ways as the envelope of a variable quadric.*

VII. If it be required to find how many double tangents can be drawn from a given point to a cyclide, let us substitute the coordinates of the given point in the quadric $S+\mu C+\mu^2$, and we shall have a quadratic in μ; hence two quadrics of each of the five systems of generating quadrics pass through the given point, and two rectilinear generators of each quadric pass through the given point; now each rectilinear generator of the generating quadric is a double tangent of the cyclide. Hence we have the following theorem:—*The tangent cone from an arbitrary point to a cyclide which has no node has twenty double edges.*

VIII. If $F\equiv(a, b, c, d, l, m, n, p, q, r \between x, y, z, 1)^2=0,$

$J\equiv x^2+y^2+z^2-r^2=0,$

the cyclide is given by the determinant

$$\begin{vmatrix} a, & n, & m, & p, & -2x, \\ n, & b, & l, & q, & -2y, \\ m, & l, & c, & r, & -2z, \\ p, & q, & r, & d, & (x^2+y^2+z^2+r^2), \\ -2x & -2y & -2z, & (x^2+y^2+z^2+r^2), & 0, \end{vmatrix}=0.\text{—January 1872.]}$$

their points of intersection. Forming the discriminant, we get

$$(1-S'')k^2+2(1-R)k+(1-S')=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

where S', S'' denote the results of substituting the coefficients of x, y, z from the planes L and M in the point sphere S $(x^2+y^2+z^2=0)$, and R the result of substituting the coefficients from one of these planes in the equation of the other. Hence if ϱ', ϱ'' be the spherical radii of the circle $S^{\frac{1}{2}}-L$ and $S^{\frac{1}{2}}-M$, we have

$$1-S'=-\tan^2\varrho',$$
$$1-S''=-\tan^2\varrho'',$$
$$1-R=-\tan\varrho'\tan\varrho''\cos C,$$

where C is the angle of intersection of the circles. Hence the quadratic (48) becomes

$$\tan^2\varrho''\,k^2+2(\tan\varrho'\tan\varrho''\cos C)k+\tan^2\varrho'=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

and the discriminant is

$$\tan^2\varrho\tan^2\varrho''\sin^2 C\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

and this is what corresponds, in the geometry of two small circles on the sphere, to the invariant of two conics,

$$(1-S')(1-S'')-(1-R)^2.$$

See SALMON'S 'Conics,' page 343, or 'Bicircular Quartics,' art. 127.

47. If D be the spherical distance between the poles of the planes L, M, we have

$$1-R=1-\frac{\cos D}{\cos\varrho'\cos\varrho''}.$$

Hence if $1-R=0$, $\cos D=\cos\varrho'\cos\varrho''$, or the triangle is right-angled which is formed by D, ϱ', ϱ'', that is the circles $S^{\frac{1}{2}}-L$, $S^{\frac{1}{2}}-M$ cut orthogonally (compare art. 9).

48. The two factors $1-R\pm\sqrt{(1-S')(1-S'')}$ of the invariant

$$(1-R)^2-(1-S')(1-S'')$$

are plainly

$$\left.\begin{array}{l}\tan\varrho'\tan\varrho''\sin\frac{1}{2}C,\\ \tan\varrho'\tan\varrho''\cos\frac{1}{2}C,\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

where C is the angle of intersection of the circles $S^{\frac{1}{2}}-L$, $S^{\frac{1}{2}}-M$; and these are respectively the sine squared of half the direct common tangent, and the sine squared of half the transverse common tangent of the two circles. We have therefore, from the extension of PTOLEMY'S theorem in my memoir "On the Equations of Circles," this further extension to conics inscribed in the same conic, namely, the condition that four conics, $S^{\frac{1}{2}}-L$, $S^{\frac{1}{2}}-M$, $S^{\frac{1}{2}}-N$, $S^{\frac{1}{2}}-P$, should be all touched by a fifth conic of the same form is

$$\sqrt{(12)(34)}\pm\sqrt{(13)(24)}\pm\sqrt{(14)(23)}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

where (12) stands for the invariant

$$(1-R)-\sqrt{(1-S')(1-S'')} \text{ of two conics.}$$

49. If we eliminate k between the equation

$$S^{\frac{1}{2}}-L+k(S^{\frac{1}{2}}-M),$$

and the discriminant,

$$k^2\tan^2\varrho''+2k\tan\varrho'\tan\varrho''\cos C+\tan^2\varrho'=0,$$

we get

$$(S^{\frac{1}{2}}-L)^2\tan^2\varrho''-2(S^{\frac{1}{2}}-L)(S^{\frac{1}{2}}-M)\tan\varrho'\tan\varrho''\cos C+(S^{\frac{1}{2}}-M)\tan^2\varrho'=0. \quad . \quad . \quad (53)$$

This is the equation of the limiting points of the two circles $S^{\frac{1}{2}}-L=0$ and $S^{\frac{1}{2}}-M=0$; and they evidently correspond to the vertices or points of intersection of the two pairs of lines which can be drawn touching the conic S through the points of intersection of the conics $S^{\frac{1}{2}}-L$ and $S^{\frac{1}{2}}-M$, with their common chord $L-M$. Compare art. 8, equations (11) and (12).

Cor. The equations of the pair of points diametrically opposite is got by changing the signs of L and M in the circles $S^{\frac{1}{2}}-L$ and $S^{\frac{1}{2}}-M$.

50. We may get the equations of the limiting points otherwise. Thus, if $\cos\alpha'$, $\cos\beta'$, $\cos\gamma'$; $\cos\alpha''$, $\cos\beta''$, $\cos\gamma''$ be the direction cosines of the planes L and M, then, when we write the equations of the small circles in the form $S^{\frac{1}{2}}-L=0$, $S^{\frac{1}{2}}-M=0$, we must have

$$L\equiv\sec\varrho'(x\cos\alpha'+y\cos\beta'+z\cos\gamma'),$$
$$M\equiv\sec\varrho''(x\cos\alpha''+y\cos\beta''+z\cos\gamma'').$$

Let, then, the circle $S^{\frac{1}{2}}-L+k(S^{\frac{1}{2}}-M)=0$ be denoted by

$$S^{\frac{1}{2}}-\sec r(x\cos\lambda+y\cos\mu+z\cos\nu)=0;$$

and if this reduce to a point, we must have $\sec r=1$.

Hence, comparing coefficients, we get

$$(1+k)\cos\lambda=\sec\varrho'\cos\alpha'+k\sec\varrho''\cos\alpha'',$$
$$(1+k)\cos\mu=\sec\varrho'\cos\beta'+k\sec\varrho''\cos\beta'',$$
$$(1+k)\cos\nu=\sec\varrho'\cos\gamma'+k\sec\varrho''\cos\gamma'';$$

square and add, and we get, after a slight reduction,

$$k^2\tan^2\varrho''+2k\tan\varrho'\tan\varrho''\cos C+\tan^2\varrho'=0,$$

the same as before.

51. We can now, from the results proved in 'Bicircular Quartics,' write out at once corresponding ones for three small circles on the sphere. Thus, from the equations of the four conics J, J', J'', J''' orthogonal to three given conics, $S-L^2$, $S-M^2$, $S-N^2=0$, we can write out the equations of the circles cutting three circles orthogonally. Thus if the circles be $S^{\frac{1}{2}}-L$, $S^{\frac{1}{2}}-M$, $S^{\frac{1}{2}}-N$, their spherical radii ϱ', ϱ'', ϱ'''; direction angles of the planes L, M, N be α', β', γ'; α'', β'', γ''; α''', β''', γ''', the orthogonal circle J is the

determinant

$$\begin{vmatrix} S^{\frac{1}{2}}, & x, & y, & z, \\ \cos\varrho', & \cos\alpha', & \cos\beta', & \cos\gamma', \\ \cos\varrho'', & \cos\alpha'', & \cos\beta'', & \cos\gamma'', \\ \cos\varrho''', & \cos\alpha''', & \cos\beta''', & \cos\gamma''', \end{vmatrix} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

Compare art 11, equation (16).

52. The foregoing equation can be got directly as follows, my object in giving the above method being to show the identity of the methods of spherical geometry, and the method of conics given in the 'Bicirculars;' and in fact it was geometrically, that is, from consideration of the sphere, that I first discovered the method given in the 'Bicirculars.'

Let π be the radius and λ, μ, ν the direction-angles of the axis of the orthogonal circle; then, from the condition $1-R=0$ (see art. 47), we get three equations,

$$\cos\varrho' \cos\pi - \cos\lambda\cos\alpha' - \cos\mu\cos\beta' - \cos\nu\cos\gamma' = 0,$$

$$\cos\varrho'' \cos\pi - \cos\lambda\cos\alpha'' - \cos\mu\cos\beta'' - \cos\nu\cos\gamma'' = 0,$$

$$\cos\varrho''' \cos\pi - \cos\lambda\cos\alpha''' - \cos\mu\cos\beta''' - \cos\nu\cos\gamma''' = 0;$$

and the required circles give us a fourth equation,

$$S^{\frac{1}{2}}\cos\pi - \cos\lambda(x) - \cos\mu(y) - \cos\nu(z) = 0.$$

Hence, eliminating linearly, we get the same determinant as before.

Cor. The equations of the three other J's are got from the equation (54) by putting negative signs to the direction cosines of the axes of the circles.

53. The equation (54) expanded is

$$S^{\frac{1}{2}}\begin{vmatrix} \cos\alpha', & \cos\beta', & \cos\gamma', \\ \cos\alpha'', & \cos\beta'', & \cos\gamma'', \\ \cos\alpha''', & \cos\beta''', & \cos\gamma''', \end{vmatrix} = \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

$$x\begin{vmatrix} \cos\beta', & \cos\gamma', & \cos\varrho', \\ \cos\beta'', & \cos\gamma'', & \cos\varrho'', \\ \cos\beta''', & \cos\gamma''', & \cos\varrho''', \end{vmatrix} + y\begin{vmatrix} \cos\gamma', & \cos\alpha', & \cos\varrho', \\ \cos\gamma'', & \cos\alpha'', & \cos\varrho'', \\ \cos\gamma''', & \cos\alpha''', & \cos\varrho''', \end{vmatrix} + z\begin{vmatrix} \cos\alpha', & \cos\beta', & \cos\varrho', \\ \cos\alpha'', & \cos\beta'', & \cos\varrho'', \\ \cos\alpha''', & \cos\beta''', & \cos\varrho'''. \end{vmatrix}$$

Let this be written $GS^{\frac{1}{2}} = Ix + Hy + Kz$, and comparing it with the equation

$$S^{\frac{1}{2}} = \sec r(x\cos\lambda + y\cos\mu + z\cos\nu),$$

we get

$$\sec^2 r = \frac{I^2 + H^2 + K^2}{G^2}.$$

Hence the coordinates of the pole of the plane of the orthogonal circle, with respect to the sphere U, are

$$\frac{I}{G}, \frac{H}{G}, \frac{K}{G}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

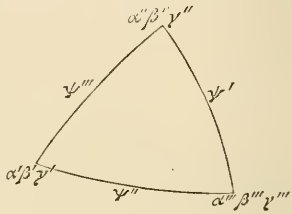
Now if this point be within the surface of the sphere U on which the circles $S^{\frac{1}{2}}-L$, $S^{\frac{1}{2}}-M$, $S^{\frac{1}{2}}-N$ are described, the orthogonal circle will be imaginary. But if the circles

$S^{\frac{1}{2}}-L$, $S^{\frac{1}{2}}-M$, $S^{\frac{1}{2}}-N$ be great circles, we have $\varrho'=\varrho''=\varrho'''=\frac{\pi}{2}$, and I, H, K each=zero, but G is finite. Hence the orthogonal circle becomes the imaginary circle at infinity. Hence we have the important theorem:—*That the imaginary circle at infinity is the circle which cuts three great circles of the sphere orthogonally, and is therefore a limiting case of the circle cutting any three circles orthogonally.*

Cor. In the geometry of a plane the two circular points at infinity are represented by the circle which cuts the three sides of any triangle orthogonally, and is therefore a limiting case of the circle which cuts any three circles on the plane orthogonally.

54. The following transformation of equation (55) will be useful in a subsequent article. Let the sides of the spherical triangle formed by joining the spherical centres of the three small circles $S^{\frac{1}{2}}-L$, $S^{\frac{1}{2}}-M$, $S^{\frac{1}{2}}-N$ be denoted by ψ', ψ'', ψ''' respectively, and the direction cosines of the planes of these sides, or, which is the same thing, of the lines from the centre of the sphere U to the angle points of the supplemental triangle be $\cos a'$, $\cos b'$, $\cos c'$; $\cos a''$, $\cos b''$, $\cos c''$; $\cos a'''$, $\cos b'''$, $\cos c'''$ respectively, then the equation (55) becomes transformed into the following:

Fig. 2.



$$S^{\frac{1}{2}}\begin{vmatrix} \cos\alpha', & \cos\beta', & \cos\gamma' \\ \cos\alpha'', & \cos\beta'', & \cos\gamma'' \\ \cos\alpha''', & \cos\beta''', & \cos\gamma''' \end{vmatrix} = \left\{\begin{array}{l} +\cos\varrho' \ \sin\psi' \ (x\cos a'+y\cos b'+z\cos c' \\ +\cos\varrho'' \ \sin\psi'' \ (x\cos a''+y\cos b''+z\cos c'' \\ +\cos\varrho''' \ \sin\psi'''(x\cos a'''+y\cos b'''+z\cos c'''. \end{array}\right\} \qquad (57)$$

55. Let us seek the locus of all the point circles of the system

$$l(S^{\frac{1}{2}}-L)+m(S^{\frac{1}{2}}-M)+n(S^{\frac{1}{2}}-N).$$

The equation $S^{\frac{1}{2}}=x\cos\lambda+y\cos\mu+z\cos\nu$ denotes a point circle (see art. 50). Hence, comparing coefficients, we get

$$\cos\lambda=l\sec\varrho'\cos\alpha'+m\sec\varrho''\cos\alpha''+n\sec\varrho'''\cos\alpha'''$$
$$\cos\mu=l\sec\varrho'\cos\beta'+m\sec\varrho''\cos\beta''+n\sec\varrho'''\cos\beta'''$$
$$\cos\nu=l\sec\varrho'\cos\gamma'+m\sec\varrho''\cos\gamma''+n\sec\varrho'''\cos\gamma''';$$

square and add, and we get

$$\left.\begin{array}{l} 1=l^2\sec^2\varrho'+m\sec^2\varrho''+n^2\sec^2\varrho'''+2lm\sec\varrho'\sec\varrho''\sin\psi''' \\ \quad +2mn\sec\varrho''\sec\varrho'''\sin\psi'+2nl\sec\varrho'''\sec\varrho'\sin\psi''. \end{array}\right\} \quad . \quad . \quad . \quad . \quad (58)$$

Now, since all that we are concerned with is the mutual ratios of the multiples l, m, n, let us suppose $l+m+n=1$; square and subtract from equation (58), and then replace l, m, n by x, y, z, and we have the equation of J (see art. 51) in the form

$$\left.\begin{array}{l} (\tan^2\varrho', \quad \tan^2\varrho'', \quad \tan^2\varrho''', \ -\tan\varrho'\tan\varrho''\cos C, \\ -\tan\varrho''\tan\varrho'''\cos A, \ -\tan\varrho'''\tan\varrho'\cos B)(x, y, z)^2=0. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (59)$$

Compare 'Bicircular Quartics,' art. 139.

56. The condition that four small circles

$$S^{\frac{1}{2}}-L,\ S^{\frac{1}{2}}-M,\ S^{\frac{1}{2}}-N,\ S^{\frac{1}{2}}-P$$

should be coorthogonal is easily seen to be the determinant

$$\begin{vmatrix} \cos\varrho', & \cos\alpha', & \cos\beta', & \cos\gamma', \\ \cos\varrho'', & \cos\alpha'', & \cos\beta'', & \cos\gamma'', \\ \cos\varrho''', & \cos\alpha''', & \cos\beta''', & \cos\gamma''', \\ \cos\varrho'''', & \cos\alpha'''', & \cos\beta'''', & \cos\gamma'''', \end{vmatrix}=0. \quad . \quad . \quad . \quad . \quad . \quad (60)$$

Now, if we form the equations of J', J'', J''' (see art. 52, *Cor.*), and using transformations similar to those of art. 54, we see that the four J's (J, J', J'', J''') fulfil the condition of being coorthogonal. Hence we have the following theorem:—*The four circles are coorthogonal which are orthogonal to the four triads of circles,* $S^{\frac{1}{2}}\pm L$, $S^{\frac{1}{2}}\pm M$, $S^{\frac{1}{2}}\pm N$.

57. The plane of the great circle coaxal with J and the small circle $S^{\frac{1}{2}}-L=0$ is the polar plane of the point whose coordinates are $\frac{I}{G}$, $\frac{H}{G}$, $\frac{K}{G}$ (see art. 53) with respect to the cone $S-L^2=0$; and this is easily found to be the determinant

$$\begin{vmatrix} L, & x, & y, & z, \\ \cos\varrho', & \cos\alpha', & \cos\beta', & \cos\gamma', \\ \cos\varrho'', & \cos\alpha'', & \cos\beta'', & \cos\gamma'', \\ \cos\varrho''', & \cos\alpha''', & \cos\beta''', & \cos\gamma''', \end{vmatrix}=0. \quad . \quad . \quad . \quad . \quad . \quad (61)$$

Compare art. 13.

58. From the condition (art. 56) that four small circles on the sphere should be coorthogonal, it is easily inferred that the poles with respect to the sphere of the planes of these circles are complanar. Hence the planes of these small circles pass through a common point. Conversely, *any four small circles on the sphere are coorthogonal whose planes pass through a common point.*

Cor. The common point through which the planes of four coorthogonal circles pass is the pole with respect to the sphere of their common Jacobian.

59. The orthogonal circle J, as will appear evident from the form of its equation (see art. 55, equation 59), has double contact with each of the four sphero-conics:

$$\frac{\cos^2\frac{1}{2}A}{x\tan\varrho'}+\frac{\cos^2\frac{1}{2}B}{y\tan\varrho''}+\frac{\cos^2\ C}{z\tan\varrho'''}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

$$\frac{\cos^2\frac{1}{2}A}{x\tan\varrho'}-\frac{\sin^2\frac{1}{2}B}{y\tan\varrho''}-\frac{\sin^2\frac{1}{2}C}{z\tan\varrho'''}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

$$\frac{-\sin^2\frac{1}{2}A}{x\tan\varrho'}+\frac{\cos^2\frac{1}{2}B}{y\tan\varrho''}-\frac{\sin^2\frac{1}{2}C}{z\tan\varrho'''}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

$$\frac{-\sin^2\frac{1}{2}A}{x\tan\varrho'}-\frac{\sin^2\ B}{y\tan\varrho''}+\frac{\cos^2\frac{1}{2}C}{z\tan\varrho'''}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

the four chords of contact being the four great circles

$$x \tan \varrho' \pm y \tan \varrho'' \pm z \tan \varrho''' = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

These great circles are the four axes of similitude of the three circles $S^{\frac{1}{2}}-L$, $S^{\frac{1}{2}}-M$, $S^{\frac{1}{2}}-N$, and the four sphero-conics are the four director or focal conics of the four pairs of circles (regarded as sphero-quartics) which are tangential to the three circles $S^{\frac{1}{2}}-L$, $S^{\frac{1}{2}}-M$, $S^{\frac{1}{2}}-N$. (See my memoir "On the Equations of Circles.")

60. Each of the four sphero-conics (62) to (65) inscribed in J touches four other sphero-conics inscribed in J; their equations are

$$J = \{x \tan \varrho' \cos(B \pm C) + y \tan^4 \varrho'' \cos(C \pm A) + z \tan \varrho''' \cos(A \pm B)^2\}. \quad . \quad (67)$$

By means of these sphero-conics can be proved Dr. HART's extension of FEUERBACH's theorem, and the equations of Dr. HART's circles can be found. (See my memoir "On the Equations of Circles.")

61. We now return, after this long digression, to the sphero-quartic.

Let us consider the function of the second degree,

$$(a, b, c, f, g, h \between S^{\frac{1}{2}}-L, S^{\frac{1}{2}}-M, S^{\frac{1}{2}}-N)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

This will represent a sphero-quartic on the surface of the sphere U, or a quartic cone having its vertex at the centre of U, according as we interpret $S^{\frac{1}{2}}-L$ &c. as circles, or as single sheets of cones (see art. 44). Now the equation (68) is the envelope of $\lambda(S^{\frac{1}{2}}-L)+\mu(S^{\frac{1}{2}}-M)+\nu(S^{\frac{1}{2}}-N)$. If the condition be fulfilled,

$$(A, B, C, F, G, H \between \lambda\ \mu\ \nu)^2 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

where A, B, &c. denote as usual $bc-f^2$, $ca-g^2$, &c.; but the circle

$$\lambda(S^{\frac{1}{2}}-L)+\mu(S^{\frac{1}{2}}-M)+\nu(S^{\frac{1}{2}}-N)=0$$

has the plane $\lambda L+\mu M+\nu N=0$ perpendicular to its axis, that is, perpendicular to the radius of U passing through its centre, and in virtue of the condition (64) the envelope of the plane is the cone

$$(a, b, c, f, g, h \between L, M, N)^2 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

and therefore the locus of the centres of the generating circles of the sphero-quartic (68) is the sphero-conic, in which the cone reciprocal to (70) intersects the sphere U.

62. If we regard the equation (68) as representing a quartic cone, we see that the locus of the axis of its generating right cone is a cone of the second degree, namely, the reciprocal of the cone (70), and its generating right cone cuts orthogonally another right cone, namely, the right cone which is orthogonal to the three cones $S^{\frac{1}{2}}-L$, $S^{\frac{1}{2}}-M$, $S^{\frac{1}{2}}-N$; the equation of the orthogonal cone is given, art. 51, equation (54).

Cor. If we suppose any plane to cut the quartic cone of this article, it will intersect it in a quartic curve having two double points; it will also intersect the generating right cone in a conic, which will be the generating conic of the quartic curve; and finally it will intersect the directing cone of the quartic cone in a conic, which will be the directing conic of the quartic curve. It was in this manner I was first led to the discovery of the

method of generating quartics having two finite double points, and the transition from that was easy to the discovery of the method of generating surfaces of the fourth degree having a conic for a double line (see Chapter II., Section II. of this paper).

63. Since the plane of each of the circles $S^{\frac{1}{2}}-L$, $S^{\frac{1}{2}}-M$, $S^{\frac{1}{2}}-N$ passes through the pole of the plane of the orthogonal circle J with respect to U, that is through the point $\left(\frac{I}{G}, \frac{H}{G}, \frac{K}{G}\right)$ (see art. 53), it follows that the plane of the generating circle

$$\lambda(S^{\frac{1}{2}}-L)+\mu(S^{\frac{1}{2}}-M)+\nu(S^{\frac{1}{2}}-N)$$

passes through the same point, therefore the pole of the plane of the generating circle with respect to U lies in the plane $Ix+Hy+Kz=G$; in other words, the pole of the generating circle is complanar with the poles of the planes of the circles of reference, and therefore it describes a conic in space, namely, the conic in which the plane $Ix+Hy+Kz=G$ intersects the cone reciprocal to

$$(a, b, c, f, g, h)(L, M, N)^2=0.$$

64. Since the planes of the circles $S^{\frac{1}{2}}-L$, $S^{\frac{1}{2}}-M$, $S^{\frac{1}{2}}-N$ are respectively parallel to the planes L, M, N, the envelope of the plane of the generating circle is a cone similar and similarly placed with the cone $(a, b, c, f, g, h)(L, M, N)^2$, and its vertex is at the point $\frac{I}{G}, \frac{H}{G}, \frac{K}{G}$. Hence we are led to the known proposition, *that a sphero-quartic is the intersection of a sphere and a cone of the second degree, and therefore that it is the intersection of a sphere and a quadric.*

65. If the poles of the planes of the generating circles $S^{\frac{1}{2}}-L$, &c. with respect to U be the centres of three spheres α, β, γ which cut the sphere U orthogonally, then α, β, γ will intersect U in the circles $S^{\frac{1}{2}}-L$, $S^{\frac{1}{2}}-M$, $S^{\frac{1}{2}}-N$ respectively, and the cyclide

$$(a, b, c, f, g, h)(\alpha, \beta, \gamma)^2$$

will intersect U in the sphero-quartic

$$(a, b, c, f, g, h)(S^{\frac{1}{2}}-L, S^{\frac{1}{2}}-M, S^{\frac{1}{2}}-N)^2.$$

Hence we are led to the known theorem, *that a sphero-quartic is the intersection of a sphere and a cyclide.*

66. The results we have arrived at in this Chapter may be projectively extended to curves described on quadrics, in other words, the analytical proof is the same for this more general case as for the particular one we have examined. Thus, instead of a sphere U of radius unity, let us take a quadric $S-K^2$ inscribed in S as the surface on which the curve is described. Now the quadric $S-L^2$ intersects $S-K^2$ in two plane conics, and we may take the equations of these two conics to $S^{\frac{1}{2}}-L=0$, and $S^{\frac{1}{2}}+L=0$, precisely similar to the method we have given of representing small circles on the surface of a sphere. It is plain that the equations $S^{\frac{1}{2}}-L$, $S^{\frac{1}{2}}+L$ have another interpretation, namely, they represent respectively the two parts into which the surface $S-L^2$ is divided by the plane $L=0$; so that on the surface of the quadric $S-K^2=0$ a plane conic is represented by an equation of the same form as that of a circle on the surface of a sphere. Again, if

we have two circles on the surface of a sphere, the condition that the pole with respect to the sphere of the plane of one may lie on the plane of the other is expressed by the invariant $1-R=0$. Hence it is evident that the same invariant relation will express for two conics on a quadric that the pole of the plane of each with respect to the quadric lies on the plane of the other; and as circles so related cut orthogonally, we shall extend the term as in art. 9, and say that two conics so related cut orthogonally. In fact the relation $1-R=0$ may be called (see art. 9) the *harmonic invariant* of the quadrics or conics whose equations are connected by it.

67. It is evident from the last article that, being given four plane conics on a quadric, the condition that the four conics should be tangential to a fifth is the determinant

$$\begin{vmatrix} 0, & (12), & (13), & (14), \\ (12), & 0, & (23), & (24), \\ (13), & (23), & 0, & (34), \\ (14), & (24), & (34), & 0, \end{vmatrix} = 0; \quad \ldots \ldots \quad (71)$$

and in fact Dr. SALMON's direct proof of this theorem in case of conics on a plane will apply *verbatim* to the more general case here considered (see SALMON's 'Conics,' 5th edition, page 366).

Cor. From the equation (71) we can find, as Dr. SALMON has done for conics on a plane, the equations of the pairs of conics which touch three conics on a quadric; the equation is

$$\sqrt{(23)(S^{\frac{1}{2}}-L)} \pm \sqrt{(31)(S^{\frac{1}{2}}-N)} \pm \sqrt{(12)(S^{\frac{1}{2}}-N)} = 0; \quad \ldots \quad (72)$$

or this equation may be inferred also from art. 59.

68. If we are given any three conics, $S^{\frac{1}{2}}-L$, $S^{\frac{1}{2}}-M$, $S^{\frac{1}{2}}-N$, on the surface of a quadric, we get, precisely the same as in art. 51, the equation of the conic J which cuts them orthogonally. And so in general, being given any homogeneous function of the second degree $(a, b, c, f, g, h \between S^{\frac{1}{2}}-L, S^{\frac{1}{2}}-M, S^{\frac{1}{2}}-N)$, we see that it represents a twisted quartic of the first family, and that all the properties of sphero-quartics may be applied projectively to it (see observations, art. 26).

CHAPTER V.—INVERSION AND CENTRES OF INVERSION.

SECTION I.—*Cyclides.*

69. If α, β, P be any three spheres, δ_1, δ_2, δ_3 their diameters, DE, FG common tangents to α, P_1; β, P respectively; then if the system be inverted from any arbitrary point, and denoting the inverse system by the same letters accented, we have (see SALMON's 'Conics,' fifth edition, page 114),

$$\frac{DE^2}{\delta_1} : \frac{FG^2}{\delta_2} = \frac{D'E'^2}{\delta_1'} : \frac{F'G'^2}{\delta_2'}. \quad \ldots \ldots \quad (73)$$

Now this result holds whatever be the magnitude of P; it will be true in the limit when P reduces to a point, in which case DE^2, FG^2 become the result of substituting the

coordinates of P in the equations of the spheres α, β respectively. Hence we have the following theorem:—

The results of substituting the coordinates of any point P in the equations of two spheres, divided respectively by the diameters of those spheres, have a ratio which is unaltered by inversion.

70. The general equation of any cyclide,

$$W=(a, b, c, d, l, m, n, p, q, r,\between\alpha, \beta, \gamma, \delta,)^2=0,$$

may be written in the form

$$W=(a\delta_1^2, b\delta_2^2, c\delta_3^2, d\delta_4^2, l\delta_2\delta_3, m\delta_3\delta_1, n\delta_1\delta_2, p\delta_1\delta_4, q\delta_2\delta_4, r\delta_3\delta_4\between\frac{\alpha}{\delta_1}, \frac{\beta}{\delta_2}, \frac{\gamma}{\delta_3}, \frac{\delta}{\delta_4})^2=0;$$

and by the last article the six ratios $\frac{\alpha}{\delta_1}:\frac{\beta}{\delta_2}:\frac{\gamma}{\delta_3}:\frac{\delta}{\delta_4}$ remain unaltered by inversion. Hence, denoting the inverses by the same letters accented, the cyclide W will be inverted into a cyclide W′ given by the equation

$$\left.\begin{aligned}W=\Big(&a\left(\frac{\delta_1'}{\delta_1}\right)^2, b\left(\frac{\delta_2'}{\delta_2}\right)^2, c\left(\frac{\delta_3'}{\delta_3}\right)^2, d\left(\frac{\delta_4'}{\delta_4}\right)^2, l\left(\frac{\delta_2'\delta_3'}{\delta_2\delta_3}\right), m\left(\frac{\delta_3'\delta_1'}{\delta_3\delta_1}\right), n\left(\frac{\delta_1'\delta_2'}{\delta_1\delta_2}\right),\\ &p\left(\frac{\delta_1'\delta_4'}{\delta_1\delta_4}\right), q\left(\frac{\delta_2'\delta_4'}{\delta_2\delta_4}\right), r\left(\frac{\delta_3'\delta_4'}{\delta_3\delta_4}\right)\between\alpha', \beta', \gamma', \delta',\Big)^2=0.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (74)$$

71. We have shown that the equation of any cyclide may be written in the form

$$a\alpha^2+b\beta^2+c\gamma^2+d\delta^2+e\varepsilon^2=0 \text{ (see art. 32)},$$

where $\alpha^2+\beta^2+\gamma^2+\delta^2+\varepsilon^2=0$ is an identical relation. I shall call this form of the equation of a cyclide the canonical form; and we see by the last article that the equation of the inverse of a cyclide given by its canonical form is also in its canonical form.

Cor. If the cyclide be of the form $a\alpha^2+b\beta^2+c\gamma^2=0$, the inverse cyclide will be of the same form, that is, the inverse of a binodal cyclide is a binodal cyclide.

72. The five spheres α, β, γ, δ, ε of the canonical form are mutually orthogonal; and if we take the centre of α for a centre of inversion, and α for the sphere of inversion, each of the five spheres will be inverted into itself. Hence the centre of α is a centre of self-inversion of the cyclide. Similarly the centres of the spheres β, γ, δ, ε are centres of self-inversion. Hence we have the following theorem:—*A cyclide is an anallagmatic surface, and the centres of the five spheres of the canonical form are its five centres of inversion.*

73. We can confirm some of the foregoing results by Cartesian methods. Thus let the spheres of reference α, β, γ, δ be expressed in rectangular coordinates, and putting s for the sum of the coefficients a, b, c, &c., we get

$$W=s(x^2+y^2+z^2)^2+U_1(x^2+y^2+z^2)+U_2=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (75)$$

where U_1 and U_2 are the general equations of the first and the second degree. Now transforming into polar coordinates by putting

$$x=\rho\cos\theta, \quad y=\rho\cos\phi, \quad z=\rho\cos\psi,$$

where

$$\cos^2\theta+\cos^2\phi+\cos^2\psi=1,$$

and then inverting by putting $\rho=\frac{k^2}{\rho'}$, and changing back again to Cartesian coordinates, we evidently get an equation of the same form, which proves that the inverse of a cyclide is a cyclide.

74. If the absolute term in the equation (75) vanishes, it is evident that the coefficient of $(x^2+y^2+z^2)^2$ in the inverse surface vanishes; in other words, the inverse surface will be a cubic cyclide, that is, a cubic surface passing through the imaginary circle at infinity. The section of this cyclide by any plane will be a circular cubic, and its focal quadric will be a paraboloid. Hence *the inverse of a cyclide from any point on the cyclide will be a cubic cyclide.* This corresponds to the theorem that the inverse of a bicircular quartic from any point on the quartic itself will be a circular cubic.

75. If $s=0$ in the general equation (75), that coefficient vanishes in the inverse surface, that is, the inverse surface will want the absolute term. Hence *if we invert a cubic cyclide from any point not on the cubic itself, the inverse surface will be a quartic cyclide passing through the origin.*

76. If in the general equation (75) not only s vanish, but also the coefficients of x, y, z in U_1 each separately vanish, that equation will represent a quadric, and the centre we invert from will be a node on the inverse surface. *Hence the inverse of a quadric will be a cyclide having the origin or centre of inversion as a node.*

The species of the node will depend on that of the quadric which is inverted.

1°. If the quadric inverted be central, let its equation be

$$\frac{(x-a)^2}{L}+\frac{(y-b)^2}{M}+\frac{(z-c)^2}{N}-1=0.$$

If this be inverted by the process of art. 73, we get

$$\left(\frac{a^2}{L}+\frac{b^2}{M}+\frac{c^2}{N}-1\right)(x^2+y^2+z^2)^2-2k^2\left(\frac{ax}{L}+\frac{by}{M}+\frac{cz}{N}\right)(x^2+y^2+z^2)+k^4\left(\frac{x^2}{L}+\frac{y^2}{M}+\frac{z^2}{N}\right)=0. \quad (76)$$

Hence *the node has the cone* $\frac{x^2}{L}+\frac{y^2}{M}+\frac{z^2}{N}$ *for a tangent cone, that is, the node is a conic node; the cone will evidently be real or imaginary according as the quadric inverted is an hyperboloid or ellipsoid.*

2°. Let the quadric inverted be non-central; let its equation be (see SALMON's 'Geometry of Three Dimensions')

$$ax^2\pm by^2+2px+2qy+2rz+d=0.$$

The process of art. 73 gives for the inverse surface

$$k^4(ax^2\pm by^2)+2k^2(px+qy+rz)(x^2+y^2+z^2)+d(x^2+y^2+z^2)^2=0. \quad . \quad . \quad (77)$$

Hence the tangent cone at the node reduces to the pair of planes $ax^2\pm by^2=0$, *and the pair of planes will be imaginary or real according as the quadric inverted represents an elliptic or a hyperbolic paraboloid.*

This species of node is called by Professor CAYLEY a *binode* (see his "Cubic Surfaces" in the Philosophical Transactions for 1869, p. 231).

3°. If $b=0$ in the equation of a non-central surface the two planes become coincident, that is, the quadric cone becomes a coincident plane pair. Professor CAYLEY calls this species of node a *unode*. Hence *the inverse of a parabolic cylinder is a unodal cyclide.*

77. If we invert a binodal cyclide, $a\alpha^2+b\beta^2+c\gamma^2$, from one of the points common to the spheres of reference α, β, γ, the spheres α, β, γ will be inverted into three planes, and *therefore the inverse of a binodal cyclide from one of the nodes will be a cone of the second degree.* Conversely, *the inverse of a cone of the second degree will be a binodal cyclide.*

This conclusion may also be inferred otherwise; for as in 1°, art. 76, the inverse of the cone

$$\frac{(x-a)^2}{\mathrm{L}}+\frac{(y-b)^2}{\mathrm{M}}+\frac{(z-c)^2}{\mathrm{N}}=0$$

is the cyclide

$$\left(\frac{a^2}{\mathrm{L}}+\frac{b^2}{\mathrm{M}}+\frac{c^2}{\mathrm{N}}\right)(x^2+y^2+z^2)^2-2k^2\left(\frac{ax}{\mathrm{L}}+\frac{by}{\mathrm{M}}+\frac{cz}{\mathrm{N}}\right)(x^2+y^2+z^2)+k^4\left(\frac{x^2}{\mathrm{L}}+\frac{y^2}{\mathrm{M}}+\frac{z^2}{\mathrm{N}}\right)=0\,; \quad (78)$$

and it is evident from the form of this equation that the origin is a conic node. Again, if we transfer the origin to the point a, b, c, it will be seen that the new origin is a conic node. *Hence the inverse of a cone has two conic nodes.*

Cor. If we invert a quadric from a point on the quadric we get a cubic cyclide.

78. If one of the spheres of inversion touch one of the focal quadrics of the cyclide at one point, the focal quadric and the sphere of inversion can be expressed by the two tangential equations

$$\left.\begin{array}{l} a\lambda^2+b\mu^2+c\nu^2+2n\nu\varrho=0, \\ a'\lambda^2+b'\mu^2+c'\nu^2+2n'\nu\varrho=0. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)$$

See CAYLEY "On Developable Surfaces of the Second Order," Cambridge and Dublin Mathematical Journal, vol. v. p. 51.

Hence it follows that the equation of the cyclide and the square of its sphere of inversion U are given by the equations

$$\left.\begin{array}{l} a\alpha^2+b\beta^2+c\gamma^2+2n\gamma\delta=0, \\ a'\alpha^2+b'\beta^2+c'\gamma^2+2n'\gamma\delta=0, \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (80)$$

where α, β are spheres of inversion of the cyclide, γ a point sphere, which is a centre of inversion of the cyclide in the sense that it is the centre of a circle which inverts the cyclide into a quadric. Hence in this case there are *four* centres of inversion, namely, the centre of U and the centres of α, β, and the point sphere γ.

The spheres of inversion are, 1°, *the sphere which inverts the quadric into a cyclide*; 2°, *the inverses of the principal planes of the quadric.*

The quadric must be either an ellipsoid or a hyperboloid; and the cyclide, which is its

inverse, will have the point of contact of the two surfaces (79) for a node. It is plain this cyclide is the pedal of a quadric. FRESNEL'S surface of elasticity is an example.

The generating spheres will be, 1°, the inverses of the three systems of spheres whose centres are in the principal planes of the quadric, and which have double contact with it; 2°, the inverses of the tangent planes of the quadric.

There will be four focal quadrics, namely, the loci of the four systems of generating spheres.

79. If one of the spheres of inversion osculate the focal quadric, the tangential equation of the focal quadric and sphere of inversion can be expressed by the system

$$\left.\begin{aligned} l(\lambda^2-2\mu\nu)+m(\mu^2-2\nu\varrho)=0,\\ l'(\lambda^2-2\mu\nu)+m'(\mu^2-2\nu\varrho)=0\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (81)$$

(see CAYLEY, *suprà*).

Hence the cyclide and the square of the sphere U may be expressed by the system of equations

$$\left.\begin{aligned} l(\alpha^2-2\beta\gamma)+m(\beta^2-2\gamma\delta)=0,\\ l'(\alpha^2-2\beta\gamma)+m'(\beta^2-2\gamma\delta)=0,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (82)$$

where β is a sphere of inversion, γ a point sphere, which is a centre of inversion in the sense that it is the centre of a sphere which inverts the cyclide into a paraboloid. Hence in this case there are only three centres of inversion, namely, the centre of U, the centre of β, and the point sphere $\gamma=0$.

The three spheres of inversion are U, β, and the sphere whose centre is γ, which inverts the paraboloid into a cyclide. The spheres U and β are the inverses of the two planes of reflection of the paraboloid.

The generating spheres will be, 1°, the inverses of the two systems of spheres whose centres are in the two planes of reflection, and which have double contact with the paraboloid; 2°, the inverses of the tangent planes of the paraboloid.

There will be three focal quadrics, namely, the loci of the centres of the three systems of generating spheres.

80. The binodal cyclide $a\alpha^2+b\beta^2+c\gamma^2=0$ will have, besides the centres of the spheres α, β, γ, which are centres of inversion, an infinite number of centres of inversion lying on the line joining the two nodes.

This is the species of cyclide that is generated when the sphere of inversion U has double contact with the focal quadric F; each point of contact will plainly be a node of the cyclide; and since the surface is the inverse of a cone, there will be, as in the case of cyclides which are the inverses of hyperboloids, four systems of generating spheres*.

* [In writing this memoir I omitted trinodal and quadrinodal cyclides. My attention was directed to this omission by Professor CAYLEY. This omission was the less excusable, inasmuch as quadrinodal cyclides were the only surfaces to which the name cyclide was applied, until M. DARBOUX extended the term (see 'Comptes Rendus,' June 7, 1869). The existence of trinodal cyclides was first proved by Professor CAYLEY in the Quar-

SECTION II.—*Sphero-quartics.*

81. The theory of the centres of inversion of sphero-quartics is in a great measure identical with that of cyclides. Thus the analogue of the fundamental theorem (art. 69) is the following. If α, β, P be any three small circles on a sphere U, r_1, r_2, r_3 their radii, t, t_1 the common tangents of α, P; β, P respectively; then, if this system be inverted from any arbitrary origin on U, by the formula

$$\tan \tfrac{1}{2}\rho \tan \tfrac{1}{2}\rho' = \text{constant}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (83)$$

ρ, ρ' being arcs drawn from the origin to a point and its inverse, we get, as in art. 69,

$$\frac{\sin^2 \frac{1}{2} t}{\tan r_1} : \frac{\sin^2 \frac{1}{2} t_1}{\tan r_2} = \frac{\sin^2 \frac{1}{2} t'}{\tan r'_1} : \frac{\sin^2 \frac{1}{2} t'_1}{\tan r'_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (84)$$

---

terly Journal, vol. x. p. 34, vol. xi. p. 15. The properties of quadrinodal cyclides were first studied by DUPIN. The principal ones will be contained in the following propositions, considered from my point of view.

I. If a quadric of revolution be inverted with respect to any point, the inverse surface will be a trinodal cyclide.

Demonstration. Let S be the quadric, L a plane of circular section, P the centre of inversion, then the sphere W, which passes through P and through the circle of intersection of S and L, will intersect S in another circle M, and will pass through a circle having OP as radius (O being the point in which a plane through P parallel to L and M is intersected by the axis of revolution of S). Now if we invert from P, the sphere W will invert into a plane V, the circles of intersection of L and M with S will invert into circles Q, R in the plane V, and the fixed circle having OP as radius will invert into the radical axis of Q and R, and the points in which Q and R meet this radical axis will be nodes of the cyclide into which S inverts; the point P will be a third node. Hence the proposition is proved.

Cor. 1. If the quadric of revolution be a cone, the inverse of its vertex will be a node of the cyclide. Hence *the inverse of a cone of revolution will be a quadrinodal cyclide.*

Cor. 2. Since a cone of revolution is the envelope of a variable sphere which touches three planes, we infer by inversion *that a quadrinodal cyclide is the envelope of a variable sphere which touches three fixed spheres.*

If in this mode of generation the points common to the three spheres be real, two of the four nodes of the cyclide will be real and two imaginary.

If the points common to the three spheres be imaginary, the four nodes of the cyclide will be imaginary. In this case it is evident the three spheres may be inverted into three spheres whose centres are collinear; now the envelope of a variable sphere which touches three spheres whose centres are collinear is evidently a ring formed by the revolution of a circle round an axis in its plane. Hence *the inverse of such a ring is a quadrinodal cyclide whose four nodes are imaginary*; in fact, a ring formed by the revolution of a circle round an axis in its plane is a quadrinodal cyclide which has two imaginary nodes at infinity.

II. The envelope of the sphere

$$(x-\alpha)^2+(y-\beta)^2+(z-a)^2=m^2\{(\alpha-c)^2+\beta^2\},$$

which cuts orthogonally the plane $z=a$, will, if α, β vary, be a cone of revolution, that is the envelope of

$$x^2+y^2+z^2-2(\alpha x+\beta y+az)+(1-m^2)(\alpha^2+\beta^2)+2m^2c\alpha+a^2-m^2c^2=0;$$

or, putting for a moment

$$(1-m^2)(\alpha^2+\beta^2)+2m^2c\alpha+a^2-m^2c^2=\Omega,$$

the envelope of

$$x^2+y^2+z^2-2(\alpha x+\beta y+az)+\Omega=0$$

is a cone of revolution. Now the sphere inverse to

$$x^2+y^2+z^2-2(\alpha x+\beta y+\alpha z)+\Omega$$

Now, reasoning as in art. 69, if P become a point, since $\sin^2 \frac{1}{2}t = \frac{\alpha}{2 \cos r_1}$ (see art. 36), we have the following theorem:—

The results of substituting the spherical coordinates of any point P *on the surface of a sphere* U *in the equations of two small circles on* U, *divided respectively by the sines of the radii of these circles, have a ratio which is unaltered by inversion.*

From this theorem it follows, as in art. 70, if $W = (a, b, c, d, l, m, n, p, q, r \between \alpha, \beta, \gamma, \delta)^2 = 0$, where α, β, γ, δ are small circles on U, be the equation of any sphero-quartic, that the

---

will evidently cut orthogonally the sphere inverse to the plane $z = a$, that is, the sphere

$$x^2 + y^2 + z^2 - \frac{2}{\Omega}(\alpha x + \beta y + az) + \frac{1}{\Omega}$$

will cut orthogonally the sphere $x^2 + y^2 + z^2 = \frac{z}{a}$,

and its envelope will be the inverse of the before-mentioned cone of revolution.

Let the coordinates of the centre of

$$x^2 + y^2 + z^2 - \frac{2}{\Omega}(\alpha x + \beta y + az) + \frac{1}{\Omega} \text{ be X, Y, Z,}$$

hence

$$\alpha = \Omega X, \quad \beta = \Omega Y, \quad a = \Omega Z;$$

$$\therefore \quad \alpha = \frac{aX}{Z}, \quad \beta = \frac{aY}{Z}.$$

Hence, restoring the value of Ω, we get

$$\alpha = \{(1 - m^2)(\alpha^2 + \beta^2) + 2m^2 c\alpha + a^2 - m^2 c^2\} X;$$

and substituting the values of α and β we get

$$(1 - m^2)a^2(X^2 + Y^2) + (a^2 - m^2 c^2)Z^2 + 2m^2 caXZ - aZ = 0,$$

or, say, $\nabla = 0$; and therefore *the quadrinodal cyclide, which is the inverse of the before-mentioned cone of revolution, is the envelope of a variable sphere whose centre moves on the quadric* $\nabla = 0$, *and which cuts orthogonally the sphere* $x^2 + y^2 + z^2 = \frac{z}{a}$.

III. The plane $Z = 0$ touches $\nabla = 0$ at an umbilic; and it is also a tangent plane to the sphere $x^2 + y^2 + z^2 = \frac{z}{a}$; and the centre of the sphere is on one of the principal axes of ∇. Hence it touches ∇ at another umbilic; but if a sphere touch a quadric at two umbilics, it touches at two others. Hence we have the following theorem:—
A quadrinodal cyclide is the envelope of a variable sphere whose centre moves on a given quadric, and which cuts orthogonally a sphere which touches the quadric at four umbilics.

Cor. We can get from the canonical form of the equation of the cyclide in terms of its five spheres of inversion the condition for four nodes. Thus, let the cyclide be

$$a\alpha^2 + b\beta^2 + c\gamma^2 + d\delta^2 + e\varepsilon^2 = 0,$$

we have

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \varepsilon^2 = 0 \text{ identically.}$$

Now if two of the coefficients a, b, c, d, e be equal to one another, for instance d and e, we get for the focal quadrics three quadrics and one conic, which must be a focal conic of the three focal quadrics. The cyclide will therefore in this case have two nodes.

If two distinct pairs of the five quantities a, b, c, d, e be equal, such as $b = c$, $d = e$, then we get for the focal quadrics one quadric and two conics, and the cyclide will have four nodes.

IV. If $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$ be an ellipsoid, then the sphere $\left(x - \frac{hk}{a}\right)^2 + y^2 + z^2 = \left(\frac{bc}{a}\right)^2$ touches it at four um-

equation of the inverse sphero-quartic will be of the same form; and in particular if the sphero-quartic be of the form $a\alpha^2+b\beta^2+c\gamma^2+d\delta^2=0$, the inverse sphero-quartic will be of the same form.

82. We have seen that the general equation of a sphero-quartic can be written in the form

$$a\alpha^2+b\beta^2+c\gamma^2+d\delta^2=0,$$

where $\alpha^2+\beta^2+\gamma^2+\delta^2=0$ is an identical relation, and the circles are mutually orthogonal.

This is also evident otherwise; for the given form contains fifteen constants, namely,

---

bilics if $h^2=a^2-b^2$, $k^2=a^2-c^2$; two of the umbilics of contact are in the plane of (xz) and are real, and two in the plane (xy) and are imaginary. Now to find the equation of the cyclide that will have the ellipsoid and sphere for focal quadric and sphere of inversion, that is, for F and J; the perpendicular let fall from the centre of J on any tangent plane to F is evidently equal to $\sqrt{a^2\cos^2\alpha+b^2\cos^2\beta+c^2\cos^2\gamma}-\frac{hk\cos\alpha}{a}$; but if O be the centre of J and T the foot of the perpendicular on the tangent plane, and if the points P, P′ be taken so that $\text{OT}^2-\text{TP}^2=\text{OT}^2-\text{TP}'^2=\left(\frac{bc}{a}\right)^2$, then P, P′ are points on the cyclide; and denoting OP by ρ and OT by p, we get

$$2p\rho=\frac{b^2c^2}{a^2}+\rho^2,$$

or,

$$2\left\{\sqrt{a^2\cos^2\alpha+b^2\cos^2\beta+c^2\cos^2\gamma}-\frac{hk\cos\alpha}{a^2}\right\}\rho=\frac{b^2c^2}{a^2}+\rho^2.$$

Hence, since $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of OP or ρ, we get the equation of the cyclide

$$2\sqrt{a^2x^2+b^2y^2+c^2z^2}=x^2+y^2+z^2+\frac{2hkx}{a}+\frac{b^2c^2}{a^2},$$

or, denoting the second side of the equation by C, $4(a^2x^2+b^2y^2+c^2z^2)=\text{C}^2$.

V. If in the equation $4(a^2x^2+b^2y^2+c^2z^2)=\text{C}^2$, we put $y=0$, $z=0$, we get

$$\left(x^2+\frac{2hkx}{a}+\frac{b^2c^2}{a^2}+2ax\right)\left(x^2+\frac{2hkx}{a}+\frac{b^2c^2}{a^2}-2ax\right)=0;$$

and the four values of x in this equation being denoted by x_1, x_2, x_3, x_4, we get $x_1+x_2+x_3+x_4=-\frac{4hk}{a}$. Hence it follows that the centre of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is the centre of the cyclide. The values of x_1, x_2, x_3, x_4 are easily seen to be given by the equations

$$x_1=\ \ a+h-k-\frac{hk}{a};\qquad x_2=\ \ a-h+k-\frac{hk}{a};$$

$$x_3=-a+h+k-\frac{hk}{a};\qquad x_4=-a-h-k-\frac{hk}{a}.$$

Hence if we had taken the centre of F for origin of coordinates in the equation of the cyclide, the four values of x would be given by the system of four equations, $\pm a\pm h\pm k=0$.

VI. The equation of the cyclide referred to the centre of F as origin is given by the following symmetrical equation:

$$(x^2+y^2+z^2)^2-2x^2(a^2+h^2+k^2)-2y^2(a^2-h^2+k^2)-2z^2(a^2+h^2-k^2)$$
$$+8ahkx+(a^4+h^4+k^4-2a^2h^2-2h^2k^2-2k^2a^2)=0.$$

The sections by the planes $y=0$, $z=0$ are evidently the two pairs of circles

$$(x+k)^2+z^2=(a+h)^2,\qquad (x-k)^2+z^2=(a-h)^2,$$

$$(x+h)^2+y^2=(a+k)^2,\qquad (x-h)^2+y^2=(a-k)^2;$$

three explicitly and twelve implicitly, each of the circles α, β, γ, δ containing three; but six equations of condition are implied by the fact of the four circles being mutually orthogonal, and the identical relation $\alpha^2+\beta^2+\gamma^2+\delta^2=0$ (for constants are incorporated) is equivalent to one condition. Hence the given form contains eight independent constants; and this is the number which determines a sphero-quartic.

83. Since the circle α is orthogonal to the circles β, γ, δ, if we take either of the points in which a diameter of U perpendicular to the plane of α, called the polar line of α (see SALMON'S 'Geometry of Three Dimensions,' art. 358), cuts U, and tan² of half the cor-

---

and the centres of these two pairs of circles are the foci respectively of the sections of the ellipsoid in whose planes they lie; also the radical axis of each pair is the line joining the pair of umbilics which lies in its plane.

VII. In the equation of the cyclide given in the last article, if we put $y=0$, $z=0$, we get

$$x^4-2x^2(a^2+h^2+k^2)+8ahkx+(a^4+h^4+k^4-2a^2h^2-2h^2k^2-2k^2a^2)=0\,;$$

and the four roots being denoted as before by x_1, x_2, x_3, x_4, we see, by the system of values given in article V., that a^2, h^2, k^2 are the roots of EULER's reducing cubic for this quartic in x; and if we denote by λ, μ, ν the roots of SIMPSON's reducing cubic for the same quartic, we get

$$\lambda=-(a^2-h^2-k^2),\quad \mu=-(a^2+k^2-h^2),\quad \nu=-(a^2+h^2-k^2).$$

Hence by these values we can write the equation of the quadrinodal cyclide in the following form, due to Professor CAYLEY, who arrived at it by a mode of investigation altogether different from that used here:

$$(y^2+z^2)^2+2x^2(y^2+z^2)+\mu y^2+\nu z^2(x-x_1)(x-x_2)(x-x_3)(x-x_4)=0.$$

VIII. The centre of similitude of the first pair of circles of article VI., that is, of the pair of circles

$$(x+k)^2+z^2=(a+h)^2,\qquad (x-k)^2+z^2=(a-h)^2,$$

is easily seen to be the point whose coordinates are $\left(\frac{ak}{h},0\right)$, that is, the middle point of the line joining two umbilics of F; or, in other words, the middle point of the line joining two nodes of the cyclide, and the centre of similitude of the other pair of circles, is the middle point of the line joining the other pair of nodes.

IX. The equation of the cyclide of article IV., that is, $4(a^2x^2+b^2y^2+c^2z^2)-C^2=0$, is the envelope of the quadric $a^2x^2+b^2y^2+c^2z^2+\mu C+\mu^2=0$, or, by restoring the value of C, of the quadric

$$(a^2+\mu)x^2+(b^2+\mu)y^2+(c^2+\mu)z^2+\frac{2\mu hk}{a}x+\mu^2+\frac{\mu b^2c^2}{a^2}\,;$$

and the condition that this should represent a cone is given by the equation

$$(a^2+\mu)(b^2+\mu)(c^2+\mu^2)\left(\mu+\frac{b^2c^2}{a^2}\right)-(b^2+\mu)(c^2+\mu)\mu\left(\frac{k^2h^2}{a^2}\right)=0.$$

This equation is satisfied by four values of μ, showing there are four cones, each having double contact with the cyclide; and the vertex of each cone is a centre of inversion. Now one value of μ is evidently $-b^2$, and the corresponding cone is

$$(a^2-b^2)x^2+(c^2-b^2)y^2-\frac{2b^2hkx}{a}+b^4\left(\frac{a^2-c^2}{a^2}\right)=0\,;$$

and when this is transferred to the centre of F as origin, it becomes

$$(hx-ak)^2-(k^2-h^2)z^2=0,$$

which represents a pair of planes whose line of intersection passes through a pair of umbilics of F; any point on this line may therefore be called a centre of inversion of the cyclide, that is, any point on the line joining a corresponding pair of nodes of the cyclide is a centre of inversion of the cyclide. In like manner, from the root

responding radius as the constant of inversion (see equation 83), then in this case each of the four circles α, β, γ, δ will be inverted into itself. Hence we have the following theorem:—

Every sphero-quartic has in general four circles and eight centres or poles of inversion.

84. If we are given a focal sphero-conic F of the sphero-quartic and its corresponding circle of inversion α, the remaining circles and centres of inversion may be constructed as follows. Through the four points in which α cuts F draw three pairs of great circles, each pair will intersect in two points diametrically opposite; the six points thus deter-

---

$\mu=-c^2$, it can be shown that the line joining the other pair of nodes is such that any point of it is a centre of inversion; and from the two remaining roots the two other centres of inversion can be found.

X. If $F=\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2z}{c}=0$ be an elliptic paraboloid, and J the sphere which touches it at the umbilics, then we find, as in article IV., the equation of the cubic cyclide with four nodes to be

$$z\left(z+\frac{a^2}{e}\right)\left(z+\frac{b^2}{e}\right)+x^2\left(z+\frac{a^2}{e}\right)+y^2\left(z+\frac{b^2}{c}\right)=0.$$

The section of this surface by the plane xz consists of a line passing through the two nodes, and a circle whose centre is the focus of the section of the paraboloid by the same plane; and the section in like manner by the plane yz consists of the line joining the two other nodes, and a circle whose centre is the focus of the parabola, which is the section of the paraboloid by the plane yz.

XI. The sections of the surface by the planes xz and yz are lines of curvature of the surface.

Domonstration. The section by the plane xz is such that the centres of the generating spheres which touch the cyclide along it lie on the section of the paraboloid made by the plane of xz; hence, taking any two consecutive points on the section, it is evident that the normals at these points will lie in the plane of the section, since they pass through the centres of the generating spheres. Hence they intersect, and the proposition is proved.

We can verify this analytically; for the differential equation of the lines of curvature of any surface (see SALMON's 'Geometry of Three Dimensions,' p. 234) is

$$\begin{vmatrix} dx, & dy, & dz, \\ L, & M, & N, \\ dL, & dM, & dN, \end{vmatrix}=0;$$

and it is easy to see that this is satisfied by the equations $x=C\left(z+\frac{b^2}{c}\right)$ and $y=C'\left(z+\frac{a^2}{e}\right)$ combined with the equation of the surface, C, C′ being any constants. Hence *any plane passing through a line joining either of the pairs of nodes will be a line of curvature.*

XII. *The inverse of a line of curvature on any surface is a line of curvature on the inverse surface.* This is easily inferred from SALMON's 'Geometry of Three Dimensions,' articles 294, 479. Now, since trinodal and quadrinodal cyclides are the inverses of quadrics of revolution, we easily infer the following theorems:—

1°. *Any section of a quartic cyclide through two nodes, either both real or both imaginary, will consist of two circles, which will be lines of curvature.*

2°. *A section of a cubic cyclide through two nodes will consist of a line and a circle, which will be lines of curvature.*

3°. *If through any point* P *on a trinodal or quadrinodal cyclide we draw two planes passing through the nodal axes, the circles which are the sections at* P *intersect orthogonally.*

4°. *Every line of curvature on a trinodal or quadrinodal cyclide consists of two circles, or a line and a circle.*

5°. *The locus of the generating spheres which touch a trinodal or quadrinodal cyclide at all the points of a line of curvature is a plane conic on the corresponding focal quadric.*—January 1872.]

mined, together with the poles of α, will be the eight centres of inversion, and the three circles cutting α orthogonally, and having the first three pairs of points as poles, will be the circles of inversion.

85. If two of the points coincide in which α cuts F, there will be then only six centres and three circles of inversion; but since the coincident circles whose centres are the point of contact of α and F reduce to a point, that point will not be a centre of inversion in the ordinary sense, that is, the centre of a circle which inverts the sphero-quartic into itself, but it will be the centre of a circle which inverts the sphero-quartic into a sphero-conic. Hence in this case there are only four ordinary centres of inversion.

If three of the points coincide, there will be then only two ordinary centres of inversion, namely, the poles of α; but the point of osculation and the point diametrically opposite to it will be the poles of a circle, which inverts the quartic into a sphero-conic, and α will be the ordinary circle of inversion.

86. The following proposition is not difficult to be proved.

If ρ be the radius vector from any origin on the surface of U to a sphero-conic and, measured in the same direction from the same origin, an arc ρ' be determined by the condition that

$$2 \tan \tfrac{1}{2} \rho' = \tan \tfrac{1}{2} \rho, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

the locus of the extremity of ρ' is a sphero-quartic.

87. Besides the method of inversion hitherto used in this section, namely by the equation (83), by which any curve on the surface of a sphere is inverted into another curve on the surface of the same sphere, there is another, the more ordinary method, by which it can be inverted into a curve on the surface of another sphere or on a plane. Thus a sphero-quartic being the intersection of a cyclide W and a sphere U, and since if both be inverted from any point in space they invert into surfaces of the same kind, we see that a sphero-quartic inverts into another sphero-quartic, or into a bicircular quartic, if the arbitrary point be taken on the surface of U.

88. Since a sphero-quartic is the intersection of a sphere and a quadric, four cones can be described through it; then it is plain that the vertices of these four cones are four points in space, which are centres of self-inversion of the quartic, that is, the quartic is an anallagmatic curve, and it has four centres of self-inversion.

89. Let us consider the sphero-quartic WU, where W is the cyclide $a\alpha^2+b\beta^2+c\gamma^2+d\delta^2$, and U its sphere of inversion given by the equation $U^2=\alpha^2+\beta^2+\gamma^2+\delta^2=0$, then the centres of α, β, γ, δ are the four vertices of the four cones through WU, that is, they are the four centres of inversion. Hence we have the following theorem:—*If*

$$W=a\alpha^2+b\beta^2+c\gamma^2+d\delta^2+e\varepsilon^2=0$$

be a cyclide given in its canonical form, then the sphero-quartic, which is the intersection of W and any of its spheres of inversion, has the centres of the four remaining spheres as centres of inversion.

90. If the arbitrary point we invert from be any point on the sphero-quartic WU

itself, then W and U invert respectively into a cubic cyclide and a plane, and hence the sphero-quartic inverts into a circular cubic. *Hence if a sphero-quartic be inverted from any point on itself it inverts into a circular cubic; and conversely, if a circular cubic be inverted from any arbitrary point in space, we get a sphero-quartic passing through the centre of inversion.*

91. If $W = a\alpha^2+b\beta^2+c\gamma^2+d\delta^2=0$, and $U^2=\alpha^2+\beta^2+\gamma^2+\delta^2=0$, and we eliminate α^2 between W and U^2, we get the cyclide $(b-a)\beta^2+(c-a)\gamma^2+(d-a)\delta^2=0$ passing through the sphero-quartic WU. But the cyclide $(b-a)\beta^2+(c-a)\gamma^2+(d-a)\delta^2=0$ is the envelope of a variable sphere whose centre moves on a conic; and inverting the curve WU from any arbitrary point on U, we get a bicircular quartic, whose generating circles will be the inverses of the circles in which the generating spheres of

$$(b-a)\beta^2+(c-a)\gamma^2+(d-a)\delta^2=0$$

intersect U, and the centres of the generating circles of the bicircular will be the points in which the lines from the origin to the centres of the generating spheres of

$$(b-a)\beta^2+(c-a)\gamma^2+(d-a)\delta^2=0$$

pierce the plane into which the sphere U inverts. Now the locus of the centres of the generating spheres of $(b-a)\beta^2+(c-a)\gamma^2+(d-a)\delta^2=0$ is one of the double lines of the developable formed by tangent planes to U along the sphero-quartic WU (see art. 42). Hence we infer the following theorem:—

If a sphero-quartic WU *be inverted into a bicircular quartic, the four cones having the point we invert from as a common vertex, and whose bases are the double lines of the developable* Σ *formed by tangent planes to* U *along the sphero-quartic* WU, *will pierce the plane of the bicircular in four conics, which will be the focal conics of the bicircular.*

92. If Z be a circle on U which osculates WU, then it is evident that the pole of the plane of Z with respect to U is a point on the cuspidal edge of Σ (see last article). Again, when we invert (WU) into a bicircular, Z will invert into an osculating circle of the bicircular. Hence we have the following theorem:—

If we invert a sphero-quartic into a bicircular, the evolute of the bicircular is the curve in which its plane is intersected by the cone whose vertex is the origin of inversion, and whose base is the cuspidal edge of the developable formed by tangent planes to U *along* WU.

93. We give in this and the following article some important properties of bicircular quartics, which follow at once from the properties we have demonstrated for sphero-quartics.

Since a sphero-quartic WU is the intersection of W and U, and WU is given by the general equation

$$WU=(a, b, c, d, l, m, n, p, q, r \between \alpha, \beta, \gamma, \delta)^2=0,$$

where $\alpha, \beta, \gamma, \delta$ are circles on U, and U^2 is given by the equation (29), art. 28, hence, *when we invert* U *into a plane, the curve* WU *will be inverted into a bicircular whose equation is* $(a, b, c, d, l, m, n, p, q, r \between \alpha, \beta, \gamma, \delta)^2=0$, *and the following relation will be an identical one on the plane into which* U *inverts*:—

$$\begin{vmatrix} -1, & \cos(\alpha\beta), & \cos(\alpha\gamma), & \cos(\alpha\delta), & \alpha\div r', \\ \cos(\beta\alpha), & -1, & \cos(\beta\gamma), & \cos(\beta\delta), & \beta\div r'', \\ \cos(\gamma\alpha), & \cos(\gamma\beta), & -1, & \cos(\gamma\delta), & \gamma\div r''', \\ \cos(\delta\alpha), & \cos(\delta\beta), & \cos(\delta\gamma), & -1, & \delta\div r'''', \\ \alpha\div r', & \beta\div r'', & \gamma\div{}''', & \delta\div r'''', & 0, \end{vmatrix} = 0. \quad . \quad . \quad . \qquad (86)$$

It hence follows that every bicircular can be expressed in the form $a\alpha^2+b\beta^2+c\gamma^2+d\delta^2=0$, where α, β, γ, δ are four circles mutually orthogonal, and that for this system of circles the relation is an identical one, $\alpha^2+\beta^2+\gamma^2+\delta^2=0$.

94. From the last article we see that the same bicircular can be written in the four following forms:

$$\left.\begin{array}{l} (d-a)\alpha^2+(d-b)\beta^2+(d-c)\gamma^2=0, \\ (a-b)\beta^2+(a-c)\gamma^2+(a-d)\delta^2=0, \\ (b-c)\gamma^2+(b-d)\delta^2+(b-a)\alpha^2, \\ (c-d)\delta^2+(c-a)\alpha^2+(c-b)\beta^2, \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (87)$$

and that consequently the tangential equations of the four focal conics of the quartic are given by the equations

$$\left.\begin{array}{l} (d-a)\lambda^2+(d-b)\mu^2+(d-c)\nu^2=0, \\ (a-b)\mu^2+(a-c)\nu^2+(a-d)\varrho^2=0, \\ (b-c)\nu^2+(b-d)\varrho^2+(d-a)\lambda^2=0, \\ (c-d)\varrho^2+(c-a)\lambda^2+(c-b)\mu^2=0. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (88)$$

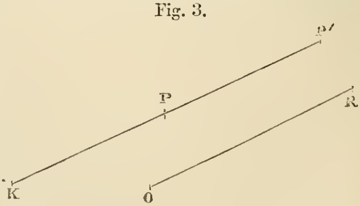
CHAPTER VI.

Projection of Sphero-quartics.

95. *If a sphero-quartic be projected on one of the planes of a circular section of any quadric passing through it by lines parallel to the greatest or least axis of the quadric, the projection will be a bicircular quartic whose centres of inversion will be the projections of the centres of inversion of the sphero-quartic.*

Demonstration. Let U be the sphere given by the equation $U^2=(\alpha^2+\beta^2+\gamma^2+\delta^2)=0$, and V the quadric which intersects U in the sphero-quartic, then the centres of α, β, γ, δ will be the vertices of the four cones which can be drawn through the sphero-quartic, that is they will be its centres of inversion. Let KPP′ be an edge of the cone passing through two points of the quartic, K being the centre of α; then, since α is a sphere of inversion of the quartic, the rectangle KP KP′ is constant. Hence if O be the centre of the quadric V, the radius vector OR of the quadric parallel to KP′ is constant; therefore the locus of R is a sphero-conic: and if the point R_1

Fig. 3.



be the projection of R on the plane of circular section, the locus of R_1 is a circle; therefore OR bears a constant ratio to its projection. Now let the projections of K, P, P′ by lines parallel to the greatest or least axis be K_1, P_1, P'_1, and it is evident that we have the proportion

$$KP \cdot KP' : K_1P_1 \cdot K_1P'_1 :: OR^2 : O_1R_1^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (89)$$

Hence the rectangle $K_1P_1 \cdot K_1P'_1$ is constant, and K_1 is a centre of inversion of the projection. The projection is therefore an anallagmatic curve, and being evidently of the fourth degree is a bicircular quartic. Hence the proposition is proved.

96. On account of the importance of the proposition of the preceding article, we give another proof by forming the equation of the projection in Cartesian coordinates.

Let U and V be given by the Cartesian equations

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2-r^2=0,$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0;$$

and changing the planes of reference to xy, xz, and one of the planes of circular section, we get, if θ denote the angle made by the plane of xy with the plane of circular section,

$$U\equiv(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2+2(x-\alpha)(z-\gamma)\cos\theta-r^2=0,$$

$$V\equiv\left(\frac{x+z\cos\theta}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z\sin\theta}{c}\right)^2-1=0.$$

If we eliminate x between these equations, we get the projection of the curve UV on the plane of circular section; this elimination is most easily performed by the following substitution, namely,

$$a^2\left(1-\frac{y^2}{b^2}-\frac{z^2\sin^2\theta}{c^2}\right)=S,$$

$$r^2-(y-\beta)^2-(z-\gamma)^2\sin^2\theta=S',$$

$$(\alpha+\gamma\cos\theta)^2=S'',$$

and we have at once the equation

$$\sqrt{S}-\sqrt{S'}+\sqrt{S''}=0,$$

or, cleared of radicals,

$$(S-S')^2+S''(S''-2S-2S')=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (90)$$

as the required projection; and substituting the value of $\sin^2\theta$, viz. $\left(\frac{a^2-b^2}{a^2-c^2}\right)\left(\frac{c^2}{b^2}\right)$ in $S-S'$, we get

$$-\left(\frac{a^2-b^2}{b^2}\right)\left\{y^2+z^2+\frac{2b^2\beta}{a^2-b^2}y+\frac{2c^2\gamma}{a^2-c^2}z-\left[\frac{a^2-\gamma^2}{a^2-c^2}b^2+\frac{b^2\beta^2}{a^2-b^2}+\frac{c^2\gamma^2}{a^2-c^2}\right]\right\},$$

which, equated to zero, represents a circle. Hence the proposition is proved.

Cor. $S+S'=-\left(\frac{a^2-b^2}{b^2}\right)\left\{\frac{a^2+b^2}{a^2-b^2}y^2+\frac{a^2+c^2}{a^2-c^2}z^2-\frac{2b^2\beta}{a^2-b^2}y-\frac{2c^2\gamma}{a^2-c^2}z+\frac{(\beta^2-a^2-\gamma^2)b^2}{a^2-b^2}+\frac{c^2\gamma^2}{a^2-c^2}\right\}.$

Hence the equation of the projection written in full, after replacing z by x, is

$$\left.\begin{array}{c}\left\{x^2+y^2+\frac{2c^2\gamma}{a^2-c^2}x+\frac{2b^2\beta}{a^2-b^2}y-\left(\frac{a^2-\gamma^2}{a^2-c^2}b^2+\frac{b^2\beta^2}{a^2-b^2}+\frac{c^2\gamma^2}{a^2-c^2}\right)\right\}^2 \\ -2\left\{\frac{\alpha b\sqrt{a^2-c^2}+\gamma a\sqrt{b^2-c^2}}{\sqrt{(a^2-b^2)(a^2-c^2)}}\right\}^2\times \\ \left\{\frac{a^2+c^2}{a^2-c^2}x^2+\frac{a^2+b^2}{a^2-b^2}y^2-\frac{2c^2\gamma}{a^2-c^2}x-\frac{2b^2\beta}{a^2-b^2}y+\frac{(\beta^2-a^2-\gamma^2)}{a^2-b^2}b^2+\frac{c^2\gamma^2}{a^2-c^2}\right\} \\ +\left\{\frac{\alpha b\sqrt{a^2-c^2}+\gamma a\sqrt{b^2-c^2}}{\sqrt{(a^2-b^2)(a^2-c^2)}}\right\}^4=0.\end{array}\right\} \quad . \quad . \quad (91)$$

97. *The elliptic projection of a sphero-quartic is a bicircular quartic.*

Definition. If through any point P on a quadric we describe two confocals, and if P′ be the point where the line of curvature common to the two confocals drawn through P intersects the plane of xy, P′ is what I call the elliptic projection of P.

If X, Y be the coordinates of the elliptic projection of a point on the sphero-quartic, x, y the coordinates of the projection of the same on either plane of circular section by lines parallel to the greatest axis, then, by SALMON's 'Geometry of Three Dimensions,' art. 180,

$$x^2 : \mathrm{X}^2 :: y^2 : \mathrm{Y}^2 :: b^2-c^2 : b^2.$$

Hence the locus of the point whose coordinates are X, Y is similar to the locus of the point whose coordinates are x, y. Hence the proposition is proved.

98. The four spheres $\alpha, \beta, \gamma, \delta$ intersect respectively the four cones through the sphero-quartic (that is, each sphere intersects the cone whose vertex is at its own centre) in four sphero-conics, and the projection of these sphero-conics on the planes of circular sections by lines parallel to the greatest or least axis will be four circles, and these will be the circles of inversion of the bicircular which results from projecting the sphero-quartic.

For if the sphere α intersect the line KPP′ in Q (see art. 95), $\mathrm{KQ}^2=\mathrm{KP}\ \mathrm{KP}'$. We can therefore account for the four circles of inversion of the bicircular.

99. The projecting lines of the four sphero-conics of the last article intersect the quadric in four curves, whose elliptic projections will be the circles of inversion of the bicircular which results from the elliptic projection of a sphero-quartic.

This is proved in the same way exactly as art. 97.

100. *Lemma.* If a sphere concentric with a quadric intersect it in a sphero-conic, and tangent planes to the quadric be parallel to the tangent planes to the cone whose vertex is at the centre of the sphere, and which stands on the sphero-conic, the locus of their points of contact is a line of curvature of the quadric.

This proposition is plainly the converse of art. 158 of SALMON's 'Geometry of Three Dimensions;' but we can give a direct proof of it as follows. Let the sphere and quadric be given by the equations

$$x^2+y^2+z^2=r^2,$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

then the equation of the cone is

$$\left(\frac{a^2-r^2}{a^2}\right)x^2+\left(\frac{b^2-r^2}{b^2}\right)y^2+\left(\frac{c^2-r^2}{c^2}\right)z^2=0.$$

Now the equation of a tangent plane to the quadric is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}+\frac{zz'}{c^2}=1.$$

Hence the equation of a parallel tangent plane through the centre is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}+\frac{zz'}{c^2}=0;$$

and the condition that this should be a tangent plane to the cone is

$$\frac{x'^2}{a^2(a^2-r^2)}+\frac{y'^2}{b^2(b^2-r^2)}+\frac{z'^2}{c^2(c^2-r^2)}=0.$$

Hence $x'\,y'\,z'$ is a point on the intersection of the confocals $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and $\frac{x^2}{a^2-r^2}+\frac{y^2}{b^2-r^2}+\frac{z^2}{c^2-r^2}=1$. Hence the proposition is proved.

101. *If tangent planes be drawn to the quadric parallel to the tangent planes of the four cones through the sphero-quartic* UV, *the locus of their points of contact are four lines of curvature on* V.

Demonstration. Let OR be a central vector of V parallel to an edge of one of the cones, then OR is constant, and the proposition is evident from the last article.

Cor. If a developable be described about V *by drawing tangent planes to it along the sphero-quartic* UV, *the four cones whose common vertex is at the centre of* V, *and which stand on the double lines of the developable, intersect the quadric in the lines of curvature stated in the proposition.*

102. *If tangent planes be drawn to* U *parallel to the tangent planes to the cones, the loci of the points of contact are sphero-conics; these sphero-conics are the focal sphero-conics of the sphero-quartic.* This proposition is evident.

103. *If through any tangent line of a sphero-quartic four planes be drawn passing through its four centres of inversion, the anharmonic ratio of these four planes is constant.*

Demonstration. Let λ_1, λ_2, λ_3, λ_4 be the four values of λ, for which $U+\lambda V$ represents a cone; then if we represent by U_1 and V_1 the tangent planes to U and V through the given line, the four planes in question are evidently tangent planes to the four cones, and their equations are $U_1+\lambda_1V_1$, $U_1+\lambda_2V_1$, $U_1+\lambda_3V_1$, $U_1+\lambda_4V_1$, and the anharmonic ratio is

$$\frac{(\lambda_1-\lambda_2)(\lambda_3-\lambda_4)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}. \qquad (92)$$

Cor. It is easy to see that the theorem, "that the anharmonic ratio is constant of the pencil formed by the four tangents which may be drawn from any point of a plane curve

of the third degree," follows as an immediate inference from the theorem of this article; for the point in the sphero-quartic will be the vertex of a cone of the third degree which stands on the quartic, and then, from what we have proved, it follows that if through any edge of a cone of the third degree four tangent planes be drawn to the cone their anharmonic ratio is constant.

CHAPTER VII.—FOCI AND FOCAL CURVES.

SECTION I.—*Foci of Cyclides.*

104. The conception of a focus which I shall use in this memoir is, in the case of surfaces, an infinitely small sphere having imaginary double contact with the surface; and for curves, that of an infinitely small circle, having imaginary double contact with the curve. This being premised, let us take the cyclide given by its canonical form,

$$W = a\alpha^2 + b\beta^2 + c\gamma^2 + d\delta^2 + e\varepsilon^2 = 0.$$

We see from art. 33 that this cyclide is the envelope in five different ways of a variable sphere whose centre moves on a given quadric, and which cuts a given fixed sphere orthogonally. Thus, taking the quadric F of art. 33, the tangential equation to F is $(a-b)\mu^2 + (a-c)\nu^2 + (a-d)\varrho^2 + (a-e)\sigma^2 = 0$; and corresponding to this we have the sphere α, which is the one which the variable sphere cuts orthogonally while its centre moves on F. Now let a developable be circumscribed to α and F, then the curve of taction of the developable and F, and the curve of intersection of α and F, divide the surface of F into three regions, which possess the following properties:—In the first region every point is such that any sphere having it as centre and cutting α orthogonally is real, and moreover such that this sphere meets the consecutive one in a real curve; in the second region every point is such that the spheres are real, but do not intersect the consecutive ones in real points; while in the third region the orthogonal spheres are altogether imaginary. Hence it follows that every point in the sphero-quartic (αF) is an infinitely small sphere having imaginary double contact with W, or, in other words, every point on (αF) is a focus of W.

In like manner every point on each of the four sphero-quartics $(\beta F')$, $(\gamma F'')$, $(\delta F''')$, (εF^{iv}) is a focus of W, so that a cyclide has in general five focal sphero-quartics.

From this proposition it is evident that the name *focal quadric* which I have employed for the directing quadrics F, F′, &c. is appropriate as suggestive of an important property of these surfaces; had I followed M. DE LA GOURNERIE I should have called them *déférentes.* In the next proposition we shall see an additional reason in favour of the name I have given.

105. *Definition.* When the points of contact of a focus with the cyclide are points on the imaginary circle at infinity, I shall, following Dr. SALMON, call the focus a "double focus" (see SALMON's 'Higher Curves'). Professor CAYLEY, in his memoir on "Polyzonal Curves," uses the term "*nodo-focus*" to express the same idea; and M. DE LA

Gournerie, in his memoir "Sur les Lignes Sphériques," "singular focus" (see Liouville's Journal for 1869).

106. Let us suppose that we have a system of generating spheres passing through the same point P; from P let there be drawn a tangent cone to the focal quadric F, then any edge of this tangent cone meets the quadric F in two consecutive points, and the generating spheres whose centres are at these points touch each other at P, consequently each edge of the cone is a normal to the cyclide at P as well as being a tangent to F: now let us suppose the point P to be on the imaginary circle at infinity, and the normals to the cyclide at P will be also tangents to it at P, and we see that the tangent lines to the cyclide at the imaginary circle at infinity are also tangent lines to the focal quadric F. Hence we have this remarkable theorem:—

The three focal conics of the focal quadric F *of the cyclide are double or nodo-focal curves of the cyclide.* Compare the corresponding theorem. art. 28 in 'Bicircular Quartics.'

107. Since the nodo-focal curves of the cyclide W are the three focal conics of F, they are in like manner the three focal conics of F', F'', F''', F^{iv}. Hence the five F's are confocal.

That is, *the five focal quadrics of a cyclide are confocal, and their three focal conics are such that each point of any of them is a double focus of the cyclide.*

108. If one of the focal quadrics of a cyclide be a sphere, then the focal conics of this sphere reduce to the centre, and the cyclide must consequently have the imaginary circle at infinity as a cuspidal edge. Hence all the focal quadrics must be spheres, and these spheres must be concentric.

From the analogy of the corresponding case in 'Bicircular Quartics' I shall call this species of cyclide a *Cartesian cyclide.*

The ordinary foci of a Cartesian cyclide being the intersection of the spheres of inversion with the focal spheres are circles, and it has but one singular or nodo-focus, which in this case is a triple focus, namely, the common centre of the focal spheres.

109. If we take four foci for spheres of reference of a cyclide, since these foci are point spheres, the result of substituting the coordinates of any point in one of them will be the square of the radius vector to the focus. Hence if we denote the vectors to the four foci by ϱ, ϱ', ϱ'', ϱ''', the vector equation of a cyclide will be in the form (see art. 29)

$$\begin{vmatrix} 0, & n, & m, & p, & \varrho^2, \\ n, & 0, & l, & q, & \varrho'^2, \\ m, & l, & 0, & r, & \varrho''^2, \\ p, & q, & r, & 0, & \varrho'''^2, \\ \varrho^2, & \varrho'^2, & \varrho''^2, & \varrho'''^2, & 0. \end{vmatrix} = 0. \qquad (93)$$

110. If the focal quadric of a cyclide be a paraboloid, then the cyclide becomes a cubic surface together with the plane at infinity. Hence *the focal quadrics of cubic cyclides are five confocal paraboloids.*

111. If we are given a sphere of inversion and the corresponding focal quadric of a cyclide, we can construct the remaining focal quadrics. For by art. 34 let α be the sphere of inversion and F the focal quadric, and circumscribing a developable Σ to α and F, if the double lines of Σ, which are conics, be C, C$'$, C$''$, C$'''$, then through C, C$'$, C$''$, C$'''$ let confocals F$'$, F$''$, F$'''$, F$''''$ to F be described, and these will be the other focal quadrics of the cyclide.

Now if α touches F the developable Σ will have but three double lines, C, C$'$, C$''$, and hence in this case there will be only four focal quadrics. If α touches F the cyclide will have the point of contact for a node, and moreover the cyclide will be the inverse of a central quadric. Hence it follows *that the cyclide which results from inverting a central quadric has but four focal quadrics, and that three of these confocal quadrics pass through the node* (see art. 78).

112. If the sphere of inversion be an osculating sphere of the focal quadric F, the developable Σ will have but two nodal lines, C, C$'$, and therefore there will be but two additional focal quadrics, F$'$, F$''$. Hence in all there will be but three focal quadrics. This is the species of cyclide which results from inverting a non-central quadric (see art. 79).

113. If the sphere of inversion has double contact with F the cyclide will be binodal; there will be, besides the focal quadric F, the two confocals to F, which can be drawn through the two points where α touches F. A cyclide of this form being given by the equation $(a, b, c, f, g, h \between \beta, \gamma, \delta)^2$, and the locus of the centre of the generating sphere being a conic, *it must be a focal conic of the three confocal quadrics which the cyclide must have, that is, every point of this conic must be a double focus of the cyclide, and moreover the four points in which it intersects the corresponding sphere of inversion must be single foci.*

When the sphere of inversion α has double contact with F, the curve of intersection of α and F breaks up into two circles; these circles are the inverses of the two focal lines of the cone, of which this species of cyclide is the inverse.

114. Since three of the spheres of inversion of a cyclide which has only four spheres of inversion, and which is consequently the inverse of a central quadric, are the inverses of the three principal planes of the quadric, and since the inverse of a focus is a focus, it follows *that in this case the inverses of the three focal conics of the quadric inverted will be the focal sphero-quartics of the cyclide.* In this case also we have the following theorem, which is an extension of one given in my 'Bicirculars,' art. 55:—

If we invert a quadric Q *from any point* P, *the principal planes of the focal quadrics of the resulting cyclide are parallel to the tangent planes at* P *drawn to three confocals of* Q *passing through* P.

115. In like manner two of the spheres of inversion of a cyclide which has only three spheres of inversion, and which is the inverse of a non-central quadric, are the inverses of the two planes of symmetry of the quadric; and since the focal conics of a paraboloid are either an ellipse and parabola or hyperbola and parabola, *we see that one of*

the focal sphero-quartics of such a cyclide must have a cusp, namely, the inverse of the point at infinity on the focal parabola.

Conversely, *if the sphere of inversion* α *of a cyclide be an osculating sphere of the focal quadric* F, *and if the whole system be inverted from the point of osculation, the sphere* α *will invert into a principal plane of the quadric into which the cyclide inverts, and the sphero-quartic in which* α *intersects* F *will invert into a parabola.*

116. Since the intersection of a focal quadric of a cyclide with the corresponding sphere of inversion gives a line of foci of the cyclide, then, if the cyclide be $a\alpha^2+b\beta^2+c\gamma^2+d\delta^2$, the focal quadric will be $a\lambda^2+b\mu^2+c\nu^2+d\varrho^2$; and if the sphere of inversion be given by the equation $U^2=\alpha^2+\beta^2+\gamma^2+\delta^2$, then the line of foci will be given as the intersector of the two surfaces in tetrahedral coordinates,

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}+\frac{w^2}{d}=0,$$

$$x^2+y^2+z^2+w^2=0,$$

and therefore the quadric in tangential coordinates

$$\frac{a\lambda^2}{a+k}+\frac{b\mu^2}{b+k}+\frac{c\nu^2}{c+k}+\frac{d\varrho^2}{d+k}=0$$

passes through the line of foci of $a\alpha^2+b\beta^2+c\gamma^2+d\delta^2$. Hence it follows that the cyclide

$$\frac{a\alpha^2}{a+k}+\frac{b\beta^2}{b+k}+\frac{c\gamma^2}{c+k}+\frac{d\delta^2}{d+k}=0 \quad \ldots \ldots \ldots \quad (94)$$

denotes in general a cyclide having one focal sphero-quartic in common with $a\alpha^2+b\beta^2+c\gamma^2+d\delta^2=0$.

117. In art. 33 we have seen that a cyclide given by the canonical form $a\alpha^2+b\beta^2+c\gamma^2+d\delta^2+e\varepsilon^2=0$ may be written in five different forms; and by the last article we see that to this cyclide correspond five different systems of cyclides, each system having one sphero-quartic of foci common with it. These systems are given by the equations:

$$\frac{(a-b)\beta^2}{a-b+k}+\frac{(a-c)\gamma^2}{a-c+k}+\frac{(a-d)\delta^2}{a-d+k}+\frac{(a-e)\varepsilon^2}{a-e+k}=0. \quad \ldots \ldots \quad (95)$$

$$\frac{(b-c)\gamma^2}{b-c+k'}+\frac{(b-d)\delta^2}{b-d+k'}+\frac{(b-e)\varepsilon^2}{b-e+k'}+\frac{(b-a)\alpha^2}{(b-a)+k'}=0. \quad \ldots \ldots \quad (96)$$

$$\frac{(c-d)\delta^2}{c-d+k''}+\frac{(c-e)\varepsilon^2}{c-e+k''}+\frac{(c-a)\alpha^2}{c-a+k''}+\frac{(c-b)\beta^2}{c-b+k''}=0. \quad \ldots \ldots \quad (97)$$

$$\frac{(d-e)\varepsilon^2}{d-e+k'''}+\frac{(d-a)\alpha^2}{d-a+k'''}+\frac{(d-b)\beta^2}{d-b+k'''}+\frac{(d-c)\gamma^2}{d-c+k'''}=0. \quad \ldots \ldots \quad (98)$$

$$\frac{(e-a)\alpha^2}{e-a+k''''}+\frac{(e-b)\beta^2}{e-b+k''''}+\frac{(e-c)\gamma^2}{e-c+k''''}+\frac{(e-d)\delta^2}{e-d+k''''}=0. \quad \ldots \ldots \quad (99)$$

In these equations the k's may have any value.

118. The method of forming the reciprocal of one cyclide with respect to another will be given in a subsequent Chapter; in this we shall anticipate so much of the results as to say that it is identical with the method of quadrics. This being premised, if we form the reciprocal of the cyclides

$$a\alpha^2+b\beta^2+c\gamma^2+d\delta^2=0,$$

$$\frac{a\alpha^2}{a+k}+\frac{b\beta^2}{b+k}+\frac{c\gamma^2}{c+k}+\frac{d\delta^2}{d+k}=0$$

with respect to $U^2\equiv\alpha^2+\beta^2+\gamma^2+\delta^2=0$, we get

$$\frac{\alpha^2}{a}+\frac{\beta^2}{b}+\frac{\gamma^2}{c}+\frac{\delta^2}{d}=0,$$

$$\frac{(a+k)\alpha^2}{a}+\frac{(b+k)\beta^2}{b}+\frac{(c+k)\gamma^2}{c}+\frac{(d+k)\delta^2}{d}=0;$$

and from the forms of these reciprocals it is plain that they have double contact along the whole sphero-quartic, in which each is intersected by the common sphere of inversion U.

119. *The three confocals to a given cyclide which can be drawn through any given point are mutually orthogonal.*

Definition. Confocal cyclides are cyclides having a common sphero-quartic of foci.

Demonstration. The focal quadrics of a confocal system of cyclides pass through a common curve of intersection; this is the sphero-quartic, which is their common line of foci. Now let P be the point through which the cyclides pass, and taking P′ the inverse of P with respect to U, then the plane which bisects P P′ perpendicularly forms with P, P′, and U a coaxial system, and the three quadrics touching this plane and passing through the common line of foci will be the focal quadrics of three confocal cyclides passing through P, P′ and cutting each other orthogonally. For if X, Y, Z be the points of contact of the quadrics with the plane, it is evident that the spheres whose centres are X, Y, Z, and which cut U orthogonally, are themselves mutually orthogonal. Hence the proposition is evident.

120. The cyclides in the last article are not only orthogonal at P, P′, but each pair of them are orthogonal throughout their whole intersection*.

Demonstration. Let us consider the two cyclides whose focal quadrics touch the plane at X, Y, and let us consider any edge of the developable which circumscribes the focal quadrics. This edge will be divided in involution by the system of quadrics passing through the common focal sphero-quartic. The double points of the involution will be the points of contact of the edge with the two quadrics which the developable circumscribes. Hence the spheres having these points as centres and cutting U orthogonally will themselves cut orthogonally, and hence it follows that the cyclides of which they are generators will cut orthogonally in a curve which has double contact with the circle of intersection of their generating spheres.

* Hence it follows, by DUPIN's theorem, that these cyclides intersect each other in lines of curvature.

Cor. Each cyclide of the three orthogonal cyclides being a surface of two sheets, hence there will be two systems, each consisting of three sheets, and each system will have eight points common to all. Hence the three orthogonal cyclides will have sixteen points common to all; these will be eight pairs of inverse points.

121. The two cyclides

$$a\alpha^2+b\beta^2+c\gamma^2+d\delta^2+e\epsilon^2,$$
$$\frac{\alpha^2}{a}+\frac{\beta^2}{b}+\frac{\gamma^2}{c}+\frac{\delta^2}{d}+\frac{\epsilon^2}{e}$$

have in common their five focal sphero-quartics.

Demonstration. For eliminate α^2 from these cyclides by means of the identical relation

$$\alpha^2+\beta^2+\gamma^2+\delta^2+\epsilon^2=0.$$

and we see, by making $k=a$ in the equation (95), that the cyclides have one focal sphero-quartic in common. Hence the proposition is proved.

122. *If two cyclides having the same spheres of inversion be reciprocals with respect to the square of any of these spheres, then each intersects this sphere in a sphero-quartic of foci of the other.*

For it is evident the cyclides

$$a\alpha^2+b\beta^2+c\gamma^2+d\delta^2$$

and

$$\frac{\alpha^2}{a}+\frac{\beta^2}{b}+\frac{\gamma^2}{c}+\frac{\delta^2}{d}$$

possess this property.

Section II.—*Foci of Sphero-quartics.*

123. We have seen that every sphero-quartic can be generated in four different ways as the envelope of a variable circle on the surface of a sphere U, the centre of the variable sphere moving along a sphero-conic while it cuts a fixed circle on U orthogonally. Now, if one of these sphero-conics be F, and α the corresponding circle, it can be seen, in the same way as in art. 104, that each point in which α intersects F is a focus of the sphero-quartic. Again, the four cones which stand on the sphero-conics (see equations (45), art. 41), and whose common vertex is at the centre of U, are plainly the reciprocals of the four cones which can be drawn through the sphero-quartic; but these latter cones have the same planes of circular section, therefore the former system have the same focal lines. Hence we have the following theorem, analogous to one in 'Bicirculars':—

Every sphero-quartic has sixteen single foci, and these lie four by four on four confocal sphero-conics, these sphero-conics being the déférentes or focal conics of the sphero-quartic.

124. Being given one circle of inversion and the corresponding focal sphero-conic of a sphero-quartic, the three remaining focal sphero-conics can be constructed. For *circumscribe a spherical quadrilateral to the circle and conic and through the three quartets of opposite intersections describe confocals, and we have the thing done.*

125. Let us consider one of the double lines of the developable which circumscribes

U along the sphero-quartic, then the cone whose vertex is at the centre of U, and which stands on this double line, intersects U in the corresponding focal sphero-conic; and it is plain that the four foci on this sphero-conic are the four points where the double line of the developable intersects U. Hence *the sixteen foci of a sphero-quartic are the sixteen points in which the double lines of the developable which circumscribes the sphere along the sphero-quartic intersect the sphere.*

126. In Chapter IV. we have shown that the equation of a sphero-quartic may be so interpreted as to represent a quartic cone, namely, by regarding the circles α, β, γ, δ, which enter into the equation of a sphero-quartic, as single sheets of a cone, whose vertex is at the centre of the sphere. Again, the equation of a sphero-quartic may be interpreted so as to represent a cyclide, that is, by regarding α, β, γ, δ as spheres cutting U orthogonally, and the quartic cone given by the former interpretation will be a tangent cone to the cyclide given by the latter. Hence we have, from article 123, the following theorem:—

The quartic cone which circumscribes a cyclide, and whose vertex is at the centre of a sphere of inversion of the cyclide, has sixteen focal lines, which are four by four the edges of four confocal cones.

127. The four confocal cones of the last article possess another important property; to demonstrate it we must prove some properties of binodal cyclides.

Let us consider the sphero-quartic WU, W being the cyclide $a\alpha^2+b\beta^2+c\gamma^2+d\delta^2$, and $U^2\equiv\alpha^2+\beta^2+\gamma^2+\delta^2$, then WU is the intersection of U with any of the four binodals got by eliminating α, β, γ, δ successively between W and U^2. Now each of these binodals has three focal quadrics and one focal conic, which focal conic is also a focal conic of the three confocal quadrics of the cyclide to which it belongs. Eliminating α, we get the binodal $(a-b)\beta^2+(a-c)\gamma^2+(a-d)\delta^2$, and the focal conic of this is one of the double lines of the developable Σ circumscribed about U along WU. Let the four double lines of Σ be the conics C, C′, C″, C‴; if C be the focal conic of $(a-b)\beta^2+(a-c)\gamma^2+(a-d)\delta^2$, then Σ is the developable circumscribed about C and U. Hence, by art. 34, the three focal quadrics will be the three quadrics described through the conics C′, C″, C‴ respectively, and having C for a focal conic.

128. Since the four cones standing on C, C′, C″, C‴ are confocal, whose vertex is at the centre of U, and the four cones are confocal which have the same vertex and of which one stands on C and the remaining three are circumscribed to the three quadrics of the last article, hence we have the theorem, *that the cones which stand on* C′, C″, C‴ *are circumscribed to the focal quadrics of the binodal which has* C *for a focal conic.*

129. From the theorems of the two last articles we infer at once the following, which is the one referred to at the commencement of art. 127:—*If* WU *be a sphero-quartic, the four cones having the centre of* U *for a common vertex, and standing on the confocal sphero-conics of* WU, *pass respectively through the focal conics of the four binodals of* WU, *and each is circumscribed to a focal quadric of each of three of these binodals.*

130. Let us denote the planes of circular section of the cones through WU by P, P′, then P passes through two of the four circular points at infinity on WU and P′ through the other two; and if Π, Π' denote the focal lines of the four cones of recent articles, we see that the tangent planes to the quartic cone Q of art. 126, which touch it at the circular points at infinity, intersect two by two in the lines Π, Π'. Hence we have the following theorem:—*The focal lines of the four confocal cones of* Q *are the double focal lines of* Q *itself.*

Cor. 1. Since every quadric has six planes of circular section, including real and imaginary, *we infer that the cone* Q *has six double focal lines.*

Cor. 2. Since the points in which the focal lines of Q intersect the sphere U are double foci of WU, *it follows that every sphero-quartic has six double foci.*

131. The theorem of the last article may be established as follows. Any plane I will cut the sphero-quartic in four points; these points are common to the four cones passing through the sphero-quartic. Hence, by reciprocation with respect to U, through any point i can be drawn four planes to intersect the planes of the nodal conics of Σ, each in four lines, forming four tetragrams described about the nodal conics. And since each tetragram has six angular points, from any point i can be drawn six lines piercing the planes of the nodal conics each in six points, which will be the angular points of tetragrams described about the nodal conics; and by supposing the plane I to be at infinity, the point i will be the centre of U, and the six lines will be the six focal lines of the four confocal cones.—Q.E.D.

132. When the circle of inversion α touches the focal sphero-conic F, the sphero-quartic has a double point, and it is the spheric inversion of a sphero-conic (see art. 81), or the ordinary inversion of a plane conic from a point outside the plane of the conic; and the cyclide WU, which will be got from the equation of the sphero-quartic by putting spheres for circles, as previously explained, will be the inversion of a central quadric. Again, the quartic cone Q, got by substituting single sheets of cones for the circle, will have a double line and three focal cones.

133. When α is an osculating circle of F, the sphero-quartic has a cusp. This species of sphero-quartic is the spheric inversion of a spherical parabola, that is, a sphero-conic whose major axis is a quadrant, or the ordinary inversion of a plane parabola from a point outside the plane of the parabola. The cyclide WU will be the inverse of a non-central quadric, and the cone Q will have a cuspidal edge, and but two confocal cones.

134. It is shown in art. 28 of 'Bicircular Quartics' that the equation of every bicircular can be written in the form $\Sigma\Sigma'=k^2C$, where Σ, Σ' are circles whose centres are the double foci of the quartic; and it is easy to see that this is equivalent to the equation in elliptic coordinates,

$$\mu^2-\nu^2=k\sqrt{C}.$$

Hence by inversion we see that any sphero-quartic can be written in the form

$$\Sigma\Sigma'=k^2C, \quad \ldots \ldots \ldots \ldots \quad (100)$$

where Σ, Σ' are small circles, whose centres are the double foci of the sphero-quartics, and that this is again equivalent to the equation in elliptic coordinates,

$$\mu^2-\nu^2=k\sqrt{C}. \quad \ldots \ldots \ldots \quad (101)$$

135. If in the equation $\Sigma\Sigma'=k^2C$ we put

$$\Sigma'=\Sigma''\pm h^2,$$

we get

$$\Sigma(\Sigma''\pm h^2)=k^2C,$$

and the intersections of the circle Σ'' with the sphero-quartic are also the points of intersection of the circle $k^2C\pm h^2\Sigma$ with the sphero-quartic. Hence the sphero-quartic meets the circle Σ'' only in two points. Hence we have the following theorem:—*Any circle whose plane is perpendicular to a double focal line of a sphero-quartic meets the sphero-quartic only in two points.*

136. Let P, P′ be the points in which the circle Σ'' meets the sphero-quartic, then PP′ will be a generator of a paraboloid passing through the quartic. Hence we easily infer the following theorem:—*If* H, H′, *the double foci of a sphero-quartic, be joined to any point* P *of the quartic, and circles described with radii* HP, H′P *cutting the sphero-quartic again in* P′, P″ *respectively, the lines* PP′, PP″ *are parallel to the planes of circular sections of quadrics passing through the quartic, and they are the generators through* P *of one of the paraboloids which can be drawn through the sphero-quartic.*

Cor. Since three paraboloids can be drawn through the sphero-quartic, this theorem affords another proof that a sphero-quartic has six double focal lines.

137. If we take the canonical form of a sphero-quartic $a\alpha^2+b\beta^2+c\gamma^2+d\delta^2=0$, we get precisely, in the same way as in art. 117, the following system of equations, each denoting a sphero-quartic confocal with the given sphero-quartic, that is, each having a quartet of foci common with it:—

$$\frac{(a-b)\beta^2}{(a-b)+k}+\frac{(a-c)\gamma^2}{(a-c)+k}+\frac{(a-d)\delta^2}{(a-d)+k}=0. \quad \ldots \ldots \quad (102)$$

$$\frac{(b-c)\gamma^2}{(b-c)+k'}+\frac{(b-d)\delta^2}{(b-d)+k'}+\frac{(b-a)\alpha^2}{(b-a)+k'}=0. \quad \ldots \ldots \quad (103)$$

$$\frac{(c-d)\delta^2}{(c-d)+k''}+\frac{(c-a)\alpha^2}{(c-a)+k''}+\frac{(c-b)\beta^2}{(c-b)+k''}=0. \quad \ldots \ldots \quad (104)$$

$$\frac{(d-a)\alpha^2}{(d-a)+k'''}+\frac{(d-b)\beta^2}{(d-b)+k'''}+\frac{(d-c)\gamma^2}{(d-c)+k'''}=0. \quad \ldots \ldots \quad (105)$$

In these equations the k's may have any value.

138. As in art. 118, we can show *that the reciprocals of two sphero-quartics having one quartet of foci common are two sphero-quartics having quartic contact at the points where they are intersected by their common circle of inversion* α.

139. *The two sphero-quartics*

$$a\alpha^2+b\beta^2+c\gamma^2+d\delta^2,$$

$$\frac{\alpha^2}{a}+\frac{\beta^2}{b}+\frac{\gamma^2}{c}+\frac{\delta^2}{d}$$

have the system of sixteen foci common to both. The proof is exactly the same as that of the corresponding theorem for two cyclides.

140. *Two sphero-quartics having four concyclic common foci can be described through any point, and they intersect orthogonally in their eight points of intersection.*

Demonstration. Let P be the given point, P′ the inverse of P with respect to the circle through the four common foci, then through the four common foci can be described two sphero-conics touching the great circle which bisects PP′ perpendicularly; these will be the focal sphero-conics of the required sphero-quartics, and the proposition is evident.

141. The construction in art. 124 may be proved as follows: from art. 131 we see that from any point can be drawn concyclic planes which will intersect the planes of the nodal conics of Σ in four tetragrams circumscribed to the nodal conics. Now if the points from which the four concyclic tangent planes are drawn be the pole of the plane of one of the nodal conics (that is, in fact, if it be one of the four centres of inversion of the sphero-quartic), the proposition is evident.

CHAPTER VIII.

Anharmonic Properties of the Developable Σ and its Reciprocal.

142. Let us consider the cyclide $W+kU^2=0$, where

$$W\equiv a\alpha^2+b\beta^2+c\gamma^2+d\delta^2,$$
$$U^2\equiv \alpha^2+\beta^2+\gamma^2+\delta^2;$$

then the tangential equation of the focal quadric of $W+kU^2$ is

$$(a+k)\lambda^2+(b+k)\mu^2+(c+k)\nu^2+(d+k)\rho^2=0,$$

and this in tetrahedral coordinates is

$$\frac{x^2}{a+k}+\frac{y^2}{b+k}+\frac{z^2}{c+k}+\frac{w^2}{d+k}=0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (106)$$

the discriminant of this with respect to k will be the developable Σ circumscribed to U along the sphero-quartic WU.

143. The differential of (106) with respect to k, gives

$$\frac{x^2}{(a+k)^2}+\frac{y^2}{(b+k)^2}+\frac{z^2}{(c+k)^2}+\frac{w^2}{(d+k)^2}=0; \quad . \quad . \quad . \quad . \quad . \quad (107)$$

and the intersection of the quadrics (106) and (107) will be the locus of the centres of the generating spheres passing through the sphero-quartic WU of the cyclide $W+kU^2$; and this curve, namely the intersection of (106) and (107), is a cuspidal edge on the surface of centres of $W+kU^2$. *Hence we see that the locus of all the cuspidal edges for all the surfaces* $W+kU^2$ *is the developable* Σ *circumscribed to* U *along* WU.

Cor. 1. The cuspidal edge of the surface of centres of any cyclide of the system $W+kU^2$ is a quartic of the first family. (See SALMON, p. 274.)

Cor. 2. These cuspidal edges have another and a more important geometrical signification; they are the curves in which the quadrics of the system

$$\frac{x^2}{(a+k)}+\frac{y^2}{(b+k)}+\frac{z^2}{(c+k)}+\frac{w^2}{(d+k)}=0$$

touch the envelope; on this account I shall call them *curves of taction.*

144. The envelope of the quadric (107) is CLEBSCH's Surface of Centres (see SALMON, page 399). If we form the tangential equation of this quadric, we get

$$(a+k)^2\lambda^2+(b+k)^2\mu^2+(c+k)^2\nu^2+(d+k)^2\varrho^2;$$

and this is the focal quadric of the cyclide

$$W'+2kW+k^2U^2=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (108)$$

where

$$W'\equiv a^2\alpha^2+b^2\beta^2+c^2\gamma^2+d^2\delta^2.$$

The envelope of (108) is the cyclide

$$W'U^2=W^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (109)$$

a surface of the eighth degree. This is the envelope of all the spheres whose centres move on CLEBSCH's Surface of Centres, and which cut a given sphere orthogonally.

145. The lines of Σ are cut homographically by its curves of taction.

Demonstration. Σ is the envelope of all the quadrics,

$$(a\lambda^2+b\mu^2+c\nu^2+d\varrho^2)+k(\lambda^2+\mu^2+\nu^2+\varrho^2);$$

and by giving four different values to k, say k', k'', &c., the anharmonic ratio of the four points in which any line of Σ is divided by the corresponding lines of taction is

$$(k'-k'')(k'''-k''''):(k'-k''')(k''-k''''). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (110)$$

Hence the proposition is proved.

A particular case is that the anharmonic ratio is constant of the four points in which any line of Σ is divided by its four nodal conics; the value of this anharmonic ratio is

$$(a-b)(c-d):(a-c)(b-d). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (111)$$

146. *The envelope of tangent lines to the curves of taction of Σ at all the points where any line of Σ meets them is a plane conic which touches the cuspidal edge.*

Let L, L′ be two consecutive lines of Σ; then, since L, L′ are divided homographically by the curves of taction, the proposition is evident.

147. If L, L′ be two non-consecutive lines of Σ, the lines joining the points where they meet the curves of taction generate a ruled quadric; this is evident, since the curves of taction divide L, L′ homographically.

Cor. The lines joining the four pairs of points in which L, L′ meet the double lines of Σ are generators of a ruled quadric.

148. The reciprocal of the developable Σ is the developable formed by the tangent lines of the sphero-quartic WU. We shall denote this latter developable by Δ.

If we reciprocate the *Cor.* in the last article we get the following theorem:—

If the four centres of inversion of WU *be joined by planes to two non-consecutive lines of* WU, *the four lines of intersection of the homologous pairs of planes are generators of a ruled quadric.*

149. *The lines of* Σ *divide its nodal conics homographically.*

Demonstration. Let five lines of Σ, namely L, L′, &c., meet its nodal conics in four systems of five points, namely l, l', &c., m, m', &c., n, n', &c., p, p', &c.; then, by art. 147, the four systems of lines

$$\begin{array}{cccc} ll' , & mm' , & nn' , & pp' , \\ ll'' , & mm'' , & nn'' , & pp'' , \\ ll''' , & mm''' , & nn''' , & pp''' , \\ ll'''' , & mm'''' , & nn'''' , & pp'''' \end{array}$$

are generators of four hyperboloids, H, H′, H″, H‴, and the planes Lll', Lll'', Lll''', Lll'''' are tangent planes to H, H′, &c.; hence the anharmonic ratios are equal, $l\{l', l'', l''', l''''\}$ $\{H\, H'\, H''\, H'''\}$. Hence the proposition is proved.

150. By reciprocating art. 149 we get the following theorem:—*If four tangent planes be drawn through any four lines of the system* WU *to one of the four cones through* WU, *the anharmonic ratio of these four tangent planes is equal to the anharmonic ratio of the four tangential planes drawn through the same lines of* WU *to any of the three remaining cones.*

Cor. From this proposition we infer the following theorem:—*The anharmonic ratio of the four edges of one of the four cones of* WU *passing through any four points on* WU *is equal to the anharmonic ratio of the four edges passing through the same points of any of the three remaining cones.*

151. Since the sphero-quartic WU is a curve of taction on Σ, the tangent line to WU at the point where L cuts it (see art. 146) is a tangent line to the conic of art. 146. This theorem may be enunciated as follows:—*A tangent plane to the sphere* U *at any point* P *of the sphero-quartic* WU *intersects the four faces of the tetrahedron whose vertices are the centres of inversion of the sphero-quartic in four lines; and the conic determined by these lines and the tangent line to* WU *at* P *will also touch the line of contact of the plane with* Σ *at the cuspidal edge of* Σ.

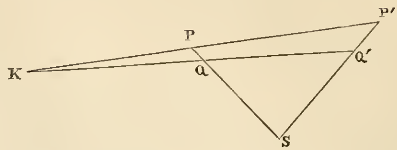
We shall have to refer so frequently to the tetrahedron formed by the centres of inversion of WU, that in order to avoid circumlocution I shall simply call it *the tetrahedron.*

152. By reciprocating with respect to U we get from the theorem of the last article this other theorem:—*The four lines drawn from any point* P *of a sphero-quartic* WU *to its four centres of inversion, the line of* Σ *passing through* P, *and the line of* Δ *through* P *are edges of the same cone of the second degree, which has also the osculating plane of* WU *at* P *for a tangent plane.*

The cone possesses this property also; namely, the anharmonic ratio is constant of the four edges passing through the centres of inversion.

153. Let us now consider the developable Δ reciprocal of Σ. This is formed by the tangent lines of WU. Let K be one of the centres of inversion of WU, P, P′, Q, Q′ two pairs of inverse points of WU; then, if P, Q be consecutive points, PQ, P′Q′ are two lines of Δ, and their point of intersection, S, is a point on two lines, and the locus of S will be a double or nodal line of Δ.

Fig. 4.



Now we have seen that, W being $a\alpha^2+b\beta^2+c\gamma^2+d\delta^2$, $U^2\equiv\alpha^2+\beta^2+\gamma^2+\delta^2$, the sphero-quartic WU will be the intersection of the quadrics

$$ax^2+by^2+cz^2+dw^2=0,$$
$$x^2+y^2+z^2+w^2=0,$$

and the equation of the nodal line of the developable Δ is (see SALMON'S 'Geometry of Three Dimensions,' art. 209)

$$\frac{(c-c)^2}{bcx^2}+\frac{(c-a)^2}{cay^2}+\frac{(a-b)^2}{abz^2}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (112)$$

The same equation may be easily inferred from 'Bicircular Quartics,' art. 43.

Hence each of the four nodal lines of Δ is a quartic curve having three double points, the double points being at the centres of inversion, which are in the plane of the nodal line and passing through the four single foci of the sphero-quartic which lie in the plane of the nodal line.

154. Every line of Δ has a corresponding line in Σ; and by art. 146 any line of Δ and the corresponding one of Σ are tangents to a conic, which also touches the four lines in which their plane intersects the faces of the tetrahedron. Hence any line of Δ, and the corresponding line of Σ, are divided homographically by the faces of the tetrahedron; but the lines of Σ are divided in a given anharmonic ratio by these faces (see art. 145). Hence *the lines of Δ are divided in a given anharmonic ratio by its four nodal lines.*

Cor. If two lines L, L′ of Δ meet its nodal lines in two systems of four points $l, m, n, p, l', m', n', p'$, the corresponding chords of the nodal lines ll', mm', nn', pp' are generators of an hyperboloid; for L, L′ are divided equianharmonically by the nodal lines of Δ.

155. *The nodal or double lines of Δ are homographic figures.*

For let five lines of Δ meet its double lines in the four systems of five points each, then are equal the four pencils

$$l\{l'\ l''\ l'''\ l''''\},\quad m\{m'\ m''\ m'''\ m''''\},$$
$$n\{n'\ n''\ n'''\ n''''\},\quad p\{p'\ p''\ p'''\ p''''\};$$

that is, the four pencils are equal which are formed by the corresponding chords of the four double lines. This follows exactly in the same way as the corresponding proposition of art. 149. Hence the proposition is proved.

156. By reciprocation we get from the *Cor.* of art. 154 the following theorem:—*If*

L, L′, L″, L‴, L^{iv} *be five lines of* Σ, *the plane joining* L *to any of the four centres of inversion will intersect the planes joining* L′, L″, *&c. to the same centre in four lines, whose anharmonic ratio will be independent of the centre used in the construction, or, in other words, will be the same for all the centres.*

157. Since the locus of all the points on two lines of Δ is a system of four plane curves, each of the fourth degree, and having three double points, it follows by reciprocation *that the envelope of all the planes through two lines of* Σ *is a system of four cones, each of the fourth class, and each cone having one of the vertices of the tetrahedron for vertex, and the three faces which meet in that vertex as double tangent planes.*

158. If L be a line of Δ, then the anharmonic ratio is constant of the pencil of planes through L to the vertices of the tetrahedron. Hence any face of the tetrahedron will intersect this pencil in four lines whose anharmonic ratio is constant. Now of the rays (four lines), three are lines from the point where L pierces the face of the tetrahedron to the three vertices in that face, and the fourth is a tangent to the cone in which the same face intersects one of the four cones through WU,—the three vertices forming a self-conjugate triangle with respect to that conic. Hence we have the following theorem :—*If from any point in one of the four nodal lines of* Δ *three lines be drawn to the three double points of that nodal line, and a fourth line be drawn tangential to the conic in which the fourth cone through* WU *pierces that face, then is constant the anharmonic ratio of the pencil thus formed.*

159. The following direct proof of the converse of the theorem of the last article, stated as a property of any quartic curve having three double points, was communicated to me by my friend J. C. MALET, Scholar of Trinity College, Dublin. *If from any point of a trinodal plane quartic three rays of a given anharmonic pencil be drawn to the nodes, the envelope of the fourth ray is a conic section.*

Let the quartic be given by the equation

$$x^2y^2+y^2z^2+z^2x^2+2(xyz)(Ax+By+Cz),$$

where (xy), (yz), (zx) are the three nodes, and let the point from which the pencil is drawn be $x'\ y'\ z'$, then three of the rays are evidently the system of determinants

$$\left\|\begin{matrix} x', & y', & z', \\ x, & y, & z, \end{matrix}\right\|$$

and these may be denoted by the concurrent systems $L=0$, $M=0$, $Lz'+My'=0$.

Now, if we denote the fourth ray of the pencil by $L+kM$, the conditions of the question give

$$\frac{y'}{z'}=\frac{k}{c}, \text{ where } c \text{ is constant;}$$

but

$$\begin{aligned} L+kM &= x(kz'-y')+yx'-kzx' \\ &= \lambda x+\mu y-\nu z \text{ suppose;} \end{aligned}$$

and by comparing coefficients we get

$$kz'-y'=\lambda,\quad x'=\mu,$$
$$kx'=\nu,\qquad kz'-cy'=0.$$

Hence we have the following values for x', y', z',

$$x'=\mu\,;\ \ y'=\frac{\lambda}{c-1}\,;\ \ z'=\frac{c\lambda\mu}{(c-1)\nu}\,;$$

and these values, substituted in the equation of the given quartic, give after reduction

$$c^2\lambda^2+c^2(c-1)^2\mu^2+(c-1)^2\nu^2+2c(c-1)(\mathrm{A}\mu\nu+\mathrm{B}\nu\lambda+\mathrm{C}c\lambda\mu)=0,$$

the tangential equation of a conic.

Cor. By reciprocation we get the following not less interesting theorem:—

If any tangent T *to a curve of the fourth class having three double tangents intersect its three double tangents in the three points* A, B, C, *and if a fourth point* D *be taken on* T, *such that the anharmonic ratio* {A B C D} *is given, the locus of* D *is a conic section.*

160. If we take any point P on one of the four nodal lines of Σ, then through P can be drawn two lines of Σ, say L, L′; let these meet the other nodal lines of Σ in the two triads of points a, a', a'', b, b', b''; then, since the lines of Σ are divided equianharmonically by its nodal lines, the two ranges are equal, P$aa'a''$, P$bb'b''$. Hence the lines are concurrent, ab, $a'b'$, $a''b''$, the point of concurrence, being the vertex of the tetrahedron opposite to the plane of the node on which is taken the point P.

161. If we denote the four nodal conics of Σ by N, N′, N″, N‴, and if J be the section of the sphere U made by the face of the tetrahedron on which N lies, P the point where a common tangent PP′ of J and N touches N, then the lines of Σ which can be drawn through P are coincident; in fact the section of Σ made by the plane of N consists of the conic N repeated twice, and of the four common tangents of J and N, the equations of the common tangents being

$$x\sqrt{\frac{b-c}{bc}}\pm y\sqrt{\frac{c-a}{ca}}\pm z\sqrt{\frac{a-b}{ab}}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (113)$$

(see SALMON's 'Geometry of Three Dimensions,' p. 161). Hence it follows from the last article that the common tangent PP′ meets each of the remaining nodal conics N′, N″, N‴, and that the tangents to N′, N″, N‴, at the points where PP′ meets them, are complanar and concurrent, the point of concurrence being the opposite vertex of the tetrahedron. Hence we easily infer the following theorem:—*The three nodal conics* N′, N″, N‴ *pass respectively through the three pairs of opposite intersections of the tetragram found by common tangents of* J *and* N. Compare art. 34, 124, and art. 38, 'Bicircular Quartics.'

162. From the theorems of this Chapter may be easily inferred properties of Bicircular Quartics; I give a couple of instances.

1°. Since the anharmonic ratio is constant of the four planes through any line of Δ to the vertices of the tetrahedron, these planes will cut the sphere U in four circles, which

four circles will belong one to each of the four systems of generating circles of the sphero-quartic WU; but if the sphero-quartic be inverted from any arbitrary point of U, it becomes a bicircular quartic. Hence the *anharmonic ratio is constant of the four generating circles of a bicircular quartic which touch each other at any point of the quartic.* See 'Bicircular Quartics,' art. 99.

2°. *If four points*, l, m, n, p, *be taken on a bicircular quartic and normals be drawn to the quartic at these points, the normals divide the focal conics of the quartic homographically.* This follows from art. 149.

163. Conversely, properties of sphero-quartics may be inferred from those of bicirculars.

If we take any line through two points E, F of the sphero-quartic WU, and through EF draw four planes each toeching WU in another point, these planes intersect U in four circles, which will become, if U be inverted into a plane, four circles intersecting a bicircular quartic in two common points and touching it, each in another point, but the anharmonic ratio of such a pencil of circles is constant (see 'Bicirculars,' art. 99). *Hence is constant the anharmonic ratio of the four planes through* EF.

Cor. 1. *If* A, B, C, D *be the four points where the planes through* EF *touch the sphero-quartic, the tangent lines to the quartic at* A, B, C, D (*that is, the lines of* Δ *through* A, B, C, D) *meet* EF *in four points whose anharmonic ratio is constant.*

Cor. 2. *The four lines of* Δ *of Cor.* 1 *are generators of a ruled quadric.*

Cor. 3. *If through the lines of* Δ *at* A, B, C, D (*that is, the four lines of Cor.* 1) *be drawn four planes intersecting the sphero-quartic in a common chord, if the common chord varies it will generate a ruled quadric.*

CHAPTER IX.

Osculating Circles of Sphero-quartics.

164. If we consider the cyclide

$$W \equiv a\alpha^2+b\beta^2+c\gamma^2+d\delta^2=0,$$

and the sphere U given by the equation

$$U^2 \equiv \alpha^2+\beta^2+\gamma^2+\delta^2=0,$$

then the quadric

$$ax^2+by^2+cz^2+dw^2,$$

which will be the reciprocal of the focal quadric of W, will pass through the sphero-quartic WU, and $x^2+y^2+z^2+w^2=0$ will be the equation of U in the same system of tetrahedral coordinates.

The section of these quadrics by the plane w will be the conic $ax^2+by^2+cz^2=0$ and the circle $x^2+y^2+z^2=0$. Now, following CLEBSCH, let us generalize the method of finding the evolute of $ax^2+by^2+cz^2$ (see SALMON'S 'Geometry of Three Dimensions,' art. 472). We have the following problem to solve, which will be the generalization of drawing a normal to a conic. Let it be required to find a point x, y, z on the conic $ax^2+by^2+cz^2$, such that the pole with respect to the circle $x^2+y^2+z^2$ of the tangent to the conic at

x, y, z shall lie on the line joining x, y, z to a given point x', y', z'; denoting the coordinates of any point on this latter line by $x'-\lambda x$, $y'-\lambda y$, $z'-\lambda z$, we find (as in SALMON, art. 472) that the generalized evolute of $ax^2+by^2+cz^2$ is the discriminant of the conic

$$\frac{ax^2}{(a+\lambda)^2}+\frac{by^2}{(b+\lambda^2)}+\frac{cz^2}{(c+\lambda)^2}=0$$

with respect to λ, and therefore the required evolute is the curve of the sixth degree

$$a^{\frac{1}{3}}(b-c)^{\frac{2}{3}}x^{\frac{2}{3}}+b^{\frac{1}{3}}(c-a)^{\frac{2}{3}}y^{\frac{2}{3}}+c^{\frac{1}{3}}(a-b)^{\frac{2}{3}}z^{\frac{2}{3}}=0\,; \quad \ldots\ldots \quad (114)$$

and the reciprocal of this with respect to the circle $x^2+y^2+z^2=0$ is the quartic curve

$$\frac{(b-c)^2}{bcx^2}+\frac{(c-a)^2}{cay^2}+\frac{(a-b)^2}{abz^2}=0. \quad \ldots\ldots\ldots \quad (115)$$

165. The equation (115) occurs so frequently in subsequent articles that we shall examine its properties with some detail. If in the equation of the developable Δ formed by the tangent lines of WU we make $w=0$, the result will be the square of (115). Hence we infer the following theorem:—*The nodal lines of the developable Δ are the reciprocals of the generalized evolute of the conics in which the reciprocals of the focal quadric are cut by the faces of the tetrahedron.*

166. If we invert the sphere U from one of the eight centres of inversion (see art. 83) into one of the faces of the tetrahedron, the sphero-quartic WU will invert into a bicircular; and it is easy to see that the nodal line of Δ in that face of the tetrahedron will be the locus of the intersection T of tangents to the bicircular at a pair of inverse points P, P′ (see art. 43, 'Bicirculars'); but the point T is evidently the centre of similitude of two consecutive generating circles of the bicircular. Hence the locus of T is the envelope of the axis of similitude of three consecutive generating circles of the bicircular. Hence we infer the following theorem:—*If a sphero-quartic* WU *be inverted into a bicircular on the plane of one of the faces of the tetrahedron, the nodal line of the developable Δ formed by the tangent lines of* WU *is the envelope of the radical axis of a pair of inverse osculating circles of the bicircular.*

167. The equation (115) may, by incorporating constants with the variables, be written in the form

$$\frac{1}{x^2}+\frac{1}{y^2}=\frac{1}{z^2}\,;$$

and in this form it will be satisfied by the coordinate of the point common to the system of determinants

$$\left\|\begin{matrix} x, & y, & z, \\ \sec\phi, & \operatorname{cosec}\phi, & 1. \end{matrix}\right\|$$

If we call this the point ϕ, then the equation of the chord joining the points ϕ, ϕ' will be the determinant

$$\left|\begin{matrix} x, & y, & z, \\ \sec\phi, & \operatorname{cosec}\phi, & 1, \\ \sec\phi', & \operatorname{cosec}\phi', & 1, \end{matrix}\right|=0. \quad \ldots\ldots \quad (116)$$

Hence we find without difficulty the tangent to be given by the equation

$$x\cos^3\varphi+y\sin^3\varphi=z; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (117)$$

and this is therefore the equation of a tangent to a nodal line of Δ.

168. *If from any point of the curve* $\frac{1}{x^2}+\frac{1}{y^2}=\frac{1}{z^2}$ *four tangents be drawn, the points of contact are in a right line.*

Demonstration. We shall simplify the proof by taking $z=$ unity. Let x' y' be the point of contact, then the tangent is

$$\frac{x}{x'^3}+\frac{y}{y'^3}=1;$$

and if $(\alpha\ \beta)$ be a point where this meets the curve again, we have the equations

$$\frac{\alpha}{x'^3}+\frac{\beta}{y'^3}=1, \quad . \quad . \quad . \quad (1)$$

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}=1, \quad . \quad . \quad . \quad (2)$$

$$\frac{1}{x'^2}+\frac{1}{y'^2}=1. \quad . \quad . \quad . \quad (3)$$

Hence from (1), (3) . . . $\frac{\alpha-x'}{x'^3}+\frac{\beta-y'}{y'^3}=0,$ . . . (4)

" " (2), (3) . . . $\frac{\alpha^2-x'^2}{\alpha^2x'^2}+\frac{\beta^2-y'^2}{\beta^2y'^2}=0,$ . . . (5)

" " (4) and (5) . . $\frac{\alpha^2}{(\alpha+x')x'}+\frac{\beta^2}{(\beta+y')y'}=0,$

or $(\beta x'-\alpha y')(\beta x'+\alpha y'-\alpha\beta)=0$. Therefore the line $\beta x'+\alpha y'-\alpha\beta=0$ passes through the points of contact, and the proposition is proved.

Cor. 1. The envelope of the line through the points of contact is a conic section; for if we seek the envelope of $\frac{x}{\alpha}+\frac{y}{\beta}=1$, subject to the condition $\frac{1}{\alpha^2}+\frac{1}{\beta^2}=1$, we get the conic section

$$x^2+y^2=1.$$

The reader is not to imagine from its form that this equation represents a circle.

Cor. 2. *The anharmonic ratio is constant of the four points in which the chord of contact meets the curve.* This follows at once by considering the pencil of four tangents from a point infinitely near the former one.

Cor. 3. *If four tangents be drawn to the evolute of a conic at the points where any tangent of the evolute meets it, these four tangents are concurrent, and the locus of their points of concurrence is a conic passing through the six cusps of the evolute.*

169. In the sphero-quartic WU, if P, P′ be inverse points with regard to one of its

spheres of inversion, (α) for instance, then the spheres orthogonal to U passing respectively through two triads of consecutive pairs of points at P, P′ will be osculating spheres of W, and their circles of intersection with U will be osculating circles of WU. The radical plane of the inverse pairs of osculating spheres will be a diametral plane of U, and will intersect the face of the tetrahedron in a line which will be a tangent line to the curve (115). Hence we have the following theorem:—*The envelope of the radical plane of a pair of inverse osculating spheres of a sphero-quartic is a cone of the fourth degree possessing the following properties:—*

1°. *It has three double edges passing through three vertices of the tetrahedron.*

2°. *It has six stationary tangent planes.*

3°. *If through any edge four tangent planes be drawn, their edges of contact are coplanar.*

4°. *The anharmonic ratio of the four edges of contact is constant.*

5°. *The envelope of the plane through the four edges of contact is a cone of the second degree touching the six stationary tangent planes.*

170. Let K be one of the vertices of the tetrahedron, and S one of the osculating circles of WU. I say the cone V, whose vertex is K and which stands on S, will have double contact with the cone whose vertex is K and which circumscribes U.

Demonstration. The cone which circumscribes U along S, and the cone whose vertex is at K and which circumscribes U, have plainly two common tangent planes; and these will evidently be tangent planes to V also. Hence the proposition is proved.

171. The cone V osculates the cone through WU having the same vertex as V. This is evident, since S passes through three consecutive points of WU. The planes of circular section of V are parallel to the plane of S, and to the plane of the inverse of S.

172. If we form the reciprocal of the cone V with respect to U, its vertex will be at the centre of U, its intersection with U will be a sphero-conic having double contact with a circle of inversion (see art. 170), (2°) osculating the corresponding focal sphero-conic (art. 171); 3°, the focal lines will pass through two points on the cuspidal edge of the developable Δ circumscribed along WU (art. 171). Hence we may ennunciate the following theorem:—*If* J *and* F *denote the two cones whose vertices are at the centre of* U, *and which stand respectively on a circle of inversion and on a focal sphero-conic of the sphero-quartic* WU, *the cone standing on the cuspidal edge of* Δ *is generated by the focal lines of a variable cone which has double contact with* J *and which osculates* F.

173. The theorem of the last article has an analogue in the theory of bicircular quartics. This may be inferred from the one for sphero-quartics; but the following is a direct proof.

First we have to find the locus of the centre of a variable circle which touches one circle and which is orthogonal to another.

Let the variable circle be

$$x^2+y^2+2gx+2fy+c=0,$$

the touched circle

$$x^2+y^2+2g'x+2f'y+c'=0, \qquad (1)$$

the orthogonal circle

$$x^2+y^2+2g''x+2f''y+c''=0. \qquad (2)$$

The given conditions supply the two equations,

$$4(g^2+f^2-c)(g'^2+f'^2-c')=(2gg'+2ff'-c-c')^2,$$
$$2gg''+2ff''-c-c''=0.$$

Hence, eliminating c, and putting x, y in place of $-g$, $-f$, which are the coordinates of the centre of the variable circle, we get for the required locus

$$4(g'^2+f'^2-c')(x^2+y^2+2gx''+2f''y+c'')=\{2(g'-g'')x+2(f'-f'')y+c'-c''\}^2, \quad (118)$$

a conic which has double contact with the circle cut orthogonally, the radical axis of the two fixed circles being the chord of contact.

The focus of the conic is the centre of the fixed circle; this is most easily seen by taking the centre of the fixed circle as origin; then $f'=0$, $g'=0$, and the equation (118) becomes that of a conic having the focus as origin, namely

$$4c'(x^2+y^2)+(2g''x+2f''y+c'+c'')^2=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (119)$$

Now if the circle (1) be an osculating circle of a bicircular quartic, and the circle (2) one of its circles of inversion J, the conic (119) must have three consecutive points common with the focal conic of the quartic which corresponds to J, namely the centres of the three generating circles of the quartic which the circle (1) touches. Hence we see that the proposition is proved, *that the evolute of a bicircular quartic is the locus of the foci of a variable conic which has double contact with a circle of inversion of the quartic, and which osculates the corresponding focal conic.*

174. The theorem proved in the last article enables us to determine the degree of the evolute of a bicircular. For let ν be CHASLES's characteristic; that is, let ν be the number of conics osculating the focal conic F of a bicircular quartic, and having double contact with the corresponding circle of inversion J, which can be described to touch a given line; then the required degree will be 3ν. Hence the degree of the evolute will be known when ν is found. We shall prove in the next article that ν is 12; therefore the degree is 36; but this number suffers a reduction, as we shall prove that it includes the line at infinity taken 24 times. Hence the reduced degree is 12.

175. To find CHASLES's characteristic ν for a system of conics osculating one given conic and having double contact with another given conic. Our solution will depend, 1°, on the question, If a variable conic touch a given line, and have double contact with a given fixed conic, to find the envelope of its chord of contact with the fixed conic.

This is solved as follows. The condition that the conic $S+P''^2$ should touch P' is the tact-invariant

$$(1+S'')S'-R^2=0 \text{ (see art. 46).}$$

Let $S=x^2+y^2+z^2$, $P''=\lambda x+\mu y+\nu z$, $P'=\lambda' x+\mu' y+\nu' z$, and the tact-invariant gives

λ, μ, ν connected by an equation of the second degree. Hence the envelope is a conic section.

2°. On the question, If a variable conic osculate one conic, and have double contact with another given conic, to find the envelope of the chord of contact.

Let the osculated conic be

$$ax^2+by^2+cz^2=0,$$

and the one of double contact

$$x^2+y^2+z^2=0,$$

then the variable conic must be of the form

$$x^2+y^2+z^2-(\lambda x+\mu y+\nu z)^2=0.$$

Now, if we want to describe a conic having double contact with $x^2+y^2+z^2$, where $\lambda x+\mu y+\nu z$ cuts it and touching $ax^2+by^2+cz^2$, the points of contact on $ax^2+by^2+cz^2$ will be given as the points of intersection of $ax^2+by^2+cz^2$ with the Jacobian of $ax^2+by^2+cz^2$, $x^2+y^2+z^2$, and $\lambda x+\mu y+\nu z$; that is, the points of contact will be the points of intersection of $ax^2+by^2+cz^2$ with the conic $\frac{\lambda(b-c)}{x}+\frac{\mu(c-a)}{y}+\frac{\nu(a-b)}{z}$; and if two of these points of intersection coincide, the conic which has double contact with $x^2+y^2+z^2$ will osculate $ax^2+by^2+cz^2$; hence we must form the condition that the conics touch

$$ax^2+by^2+cz^2=0,\quad \frac{\lambda(b-c)}{x}+\frac{\mu(c-a)}{y}+\frac{\nu(a-b)}{z}=0.$$

This is easily found to be

$$a^{\frac{1}{3}}(b-c)^{\frac{2}{3}}\lambda^{\frac{2}{3}}+b^{\frac{1}{3}}(c-a)^{\frac{2}{3}}\mu^{\frac{2}{3}}+c^{\frac{1}{3}}(a-b)^{\frac{2}{3}}\nu^{\frac{2}{3}}=0. \quad \ldots\ldots \quad (120)$$

Now, since this denotes a curve of the sixth class, and the former condition 1° a curve of the second, they will have twelve common tangents; hence $\nu=12$.

Cor. In the same way it may be proved that $\mu=12$.

176. We shall now return from our digression on bicirculars.

At the points where the nodal conic N of the developable Σ (see art. 161) cuts J, the osculating circle of the sphero-quartic WU cuts J orthogonally; and hence it is its own inverse with respect to the sphere α. Therefore the four points in which J cuts N are points of stationary osculation. Hence there are on a sphero-quartic sixteen points of stationary osculation.

Cor. The cone of articles 170, 171 in this case breaks up into two planes; and since the poles of the planes of the osculating circles of WU form the cuspidal edge of the developable Σ, *we see that* Σ *has sixteen stationary points which lie four by four on the four nodal conics* N, N′, N″, N‴, *the four stationary points on* N *being the four points of contact of the common tangents of* N *and* J*; and similarly for* N′, N″, N‴.

177. The sphero-quartic (WU) is the intersection of the two surfaces in tetrahedral coordinates,

$$ax^2+by^2+cz^2+dw^2=0,$$
$$x^2+y^2+z^2+w^2,$$

the first being the reciprocal of the focal quadric of W, and the second the sphere U. Now the osculating plane of WU at any point x', y', z', w' is (see SALMON's 'Geometry of Three Dimensions,' p. 291)

$$(a-b)(a-c)(a-d)x'^3x+(b-a)(b-c)(b-d)y'^3y$$
$$+(c-a)(c-b)(c-d)z'^3z+(d-a)(d-b)(d-c)z'^3w=0.$$

This may be written in a simpler manner: thus, if $\psi(\lambda)$ denotes a biquadratic whose roots are a, b, c, d, the coefficients of the above equation denote the results of substituting the roots a, b, c, d respectively in $\psi'(\lambda)$, so that the equation becomes

$$\psi'(a)x'^3x+\psi'(b)y'^3y+\psi'(c)z'^3z+\psi'(d)w'^3w=0. \quad . \quad . \quad . \quad . \quad . \quad (121)$$

Hence through any point can be drawn twelve planes to osculate a sphero-quartic.

Cor. 1. *Through any point on the sphere* U *can be described twelve osculating circles of* WU. *Hence* CHASLES's *characteristic* μ *for the osculating circles of a sphero-quartic is* $\mu=12$.

Cor. 2. *If the point be on the sphero-quartic itself,* $\mu=9$.

Cor. 3. *Every sphero-quartic is osculated by twelve great circles*; for twelve osculating planes can be drawn through the centre of U.

Cor. 4. Let us consider any small circle Z on the surface of U; then, since through the pole of the plane of Z can be drawn twelve planes osculating WU, we have the theorem that *any circle on the surface of* U *is cut orthogonally by twelve osculating circles of* WU.

Cor. 5. By inversion we get the following theorem for bicirculars:—*Any circle in the plane of a bicircular is cut orthogonally by twelve of its osculating circles.*

Cor. 6. The theorems that a bicircular quartic has twelve, and that a circular cubic has nine points of inflection, are the inversions of *Cors.* 1, 2.

178. Since the cuspidal edge of Σ is the locus of the poles of the osculating planes of WU, it is plain that the cone whose vertex is any point of the cuspidal edge, and which circumscribes U, will touch U along an osculating circle of WU, and that it will be an osculating right cone of the cuspidal edge (see SALMON's 'Geometry of Three Dimensions,' art. 363). Again, since twelve osculating planes of WU pass through any point, we see that the cuspidal edge is of the twelfth degree. This latter part corresponds to the theorem that the evolute of a bicircular quartic is of the twelfth degree.

179. Since the cuspidal edge is of the twelfth degree, any quadric will cut it in 24 points. Hence any cone will in general cut it in 24 points. If the cone circumscribe U, we have, by reciprocation, the theorem that *any circle on the surface of* U *touches in general* 24 *osculating circles of* WU.

Cor. 1. By inversion we get the theorem that *any circle in the plane of a bicircular is in general touched by* 24 *osculating circles of the bicircular.*

Cor. 2. *Any line in the plane of a bicircular is in general touched by* 24 *of its osculating circles.*

Cor. 3. The line at infinity being touched by 24 osculating circles, shows that *the line at infinity is counted* 24 *times* in the evolute of a bicircular (see art. 174).

Cor. 4. CHASLES's *characteristics for the osculating circles of a bicircular quartic are* $\mu=12$, $\nu=24$.

SECTION II.—*Locus of the Poles of the Osculating Circles of a Sphero-quartic.*

180. The equation (121) is the osculating plane of WU at the point $x'y'z'w'$; and if the coordinates of the pole of this plane with respect to U be X, Y, Z, W, we get

$$x'=\left(\frac{\mathrm{X}}{\psi'(a)}\right)^{\frac{1}{3}},\quad y'=\left(\frac{\mathrm{Y}}{\psi'(b)}\right)^{\frac{1}{3}},\ \&\text{c.};$$

but x', y', z', w' satisfy the two equations

$$ax^2+by^2+cz^2+dw^2=0,\quad x^2+y^2+z^2+w^2=0.$$

Hence, by substitution and replacing X, Y, Z, W by x, y, z, w, we see that the locus of the poles of the osculating circles, or, what is equivalent, that the cuspidal edge of Σ is the intersection of the two surfaces

$$\left(\frac{x}{\psi'(a)}\right)^{\frac{2}{3}}+\left(\frac{y}{\psi'(b)}\right)^{\frac{2}{3}}+\left(\frac{z}{\psi'(c)}\right)^{\frac{2}{3}}+\left(\frac{w}{\psi'(d)}\right)^{\frac{2}{3}}=0, \quad . \ . \ . \ . \ . \quad (122)$$

$$a\left(\frac{x}{\psi'(a)}\right)^{\frac{2}{3}}+b\left(\frac{y}{\psi'(b)}\right)^{\frac{2}{3}}+c\left(\frac{z}{\psi'(c)}\right)^{\frac{2}{3}}+d\left(\frac{w}{\psi'(d)}\right)^{\frac{2}{3}}=0. \quad . \ . \ . \ . \quad (123)$$

181. Since the equation (121) of the osculating plane is satisfied by the coordinates of any point in it, we must have

$$\psi'(a)x'^4+\psi'(b)y'^4+\psi'(c)z'^4+\psi'(d)w'^4=0;$$

and substituting as in the last article, we see that the cuspidal edge of Σ is a curve on the surface

$$\left\{\frac{x^4}{\psi'(a)}\right\}^{\frac{1}{3}}+\left\{\frac{y^4}{\psi'(b)}\right\}^{\frac{1}{3}}+\left\{\frac{z^4}{\psi'(c)}\right\}^{\frac{1}{3}}+\left\{\frac{w^4}{\psi'(d)}\right\}^{\frac{1}{3}}=0. \quad . \ . \ . \ . \ . \ . \ . \quad (124)$$

Or we may prove this theorem otherwise. The developable Σ is the envelope of the quadric

$$\frac{x^2}{a+k}+\frac{y^2}{b+k}+\frac{z^2}{c+k}+\frac{w^2}{d+k}=0 \text{ (see art. 142).}$$

Hence the coordinates of any point on the cuspidal edge must satisfy the system of equations:

$$\frac{x^2}{(a+k)^3}+\frac{y^2}{(b+k)^3}+\frac{z^2}{(c+k)^3}+\frac{w^2}{(d+k)^3}=0, \quad \text{. . . . . . .} \quad (125)$$

$$\frac{ax^2}{(a+k)^3}+\frac{by^2}{(b+k)^3}+\frac{cz^2}{(c+k)^3}+\frac{dw^2}{(d+k)^3}=0, \quad \text{. . . . . . .} \quad (126)$$

$$\frac{a^2x^2}{(a+k)^3}+\frac{b^2y^2}{(b+k)^3}+\frac{c^2z^2}{(c+k)^3}+\frac{d^2w^2}{(d+k)^3}=0. \quad \text{. . . . . . .} \quad (127)$$

Hence

$$\frac{x^2}{(a+k)^3}:\frac{y^2}{(b+k)^3}:\frac{z^2}{(c+k)^3}:\frac{w^2}{(d+k)^3}::\frac{1}{\psi'(a)}:\frac{1}{\psi'(b)}:\frac{1}{\psi'(c)}:\frac{1}{\psi'(d)}; \quad \text{. . . . .} \quad (128)$$

and substituting the values of $(a+k)$, $(b+k)$, &c. from these equations in

$$\frac{x^2}{a+k}+\frac{y^2}{b+k}+\text{\&c.},$$

we get the equation (124). Hence &c.

Cor. 1. By giving k any particular value, we see from the equations (125), (126), (127), *that the points on the cuspidal edge of* Σ *are, eight by eight, the points of intersection of three quadrics.*

Cor. 2. From equations (126), (127) we see that the cuspidal edge is a curve on CLEBSCH's surface of centres; and from equation (109) it follows that *the sphero-quartic* WU *is a double line on the surface which has* CLEBSCH's *surface of centres for a déférente.*

182. By eliminating k from any three of the four equations (128), we get the equations of four cones standing on the cuspidal edge. Thus one of the cones is

$$(b-c)\{\psi'(a)x^2\}^{\frac{1}{3}}+(c-a)\{\psi'(b)y^2\}^{\frac{1}{3}}+(a-b)\{\psi'(c)z^2\}^{\frac{1}{3}}=0. \quad \text{. . . .} \quad (129)$$

The vertex of this cone is one of the vertices of the tetrahedron; it possesses several properties. The following are some of the most important:—

1°. *It intersects the opposite face of the tetrahedron in* CLEBSCH's *evolute of a conic* (see art. 164).

2°. *It is the reciprocal of the corresponding double line of* Δ*—that is, of the developable formed by the tangent lines of* WU.

3°. *Every edge of it is a line through two points of the cuspidal edge of* Σ.

4°. *Every tangent plane to it is a plane through two lines of* Σ, *and it is therefore one of the four cones which are the envelopes of all the planes through two lines of that developable.*

5°. The equation, cleared of radicals, is of the form

$$\{A^2+B^2+C^2\}^3=27A^2B^2C^2. \quad \text{. . . . .} \quad (130)$$

Hence it has six cuspidal edges lying on the cone of the second degree, $A^2+B^2+C^2=0$.

6°. *Any tangent plane to it will intersect it in a pencil of four lines whose anharmonic ratio is constant.*

7°. *The tangent planes, touching it along the lines of intersection of any tangent plane,*

pass through a common line. This common line is an edge of the cone $A^2+B^2+C^2=0$ *passing through the six cuspidal edges.*

183. Let us consider an edge of the cone (129). It pierces U in two points; these are the limiting points of two inverse osculating circles of the sphero-quartic WU. The equation of the locus of these limiting points is easily found; for the tangential equation of the nodal line of Σ is the equation got by substituting λ, μ, ν in place of x, y, z in the equation of the cone. Hence, if α, β, γ be the three circles of inversion of WU, the poles of whose planes are at the three remaining vertices of the tetrahedron, the equation of the required locus will be got by substituting in the equation (129) α, β, γ for x, y, z, and therefore it will be

$$(b-c)\{\psi'(a)\alpha^2\}^{\frac{1}{3}}+(c-a)\{\psi'(b)\beta^2\}^{\frac{1}{3}}+(a-b)\{\psi'(c)\gamma^2\}^{\frac{1}{3}}=0, \quad . \quad . \quad . \quad (131)$$

a curve which has twenty four cusps.

184. If x', y', z', w' be the coordinates of any point in the sphero-quartic WU, then it follows from equation (121), combined with art. 36, that the equation of the osculating circle of WU at the point x' y' z' w' is

$$\psi'(a)x'^3(\alpha)+\psi'(b)(y'^3)(\beta)+\psi'(c)z'^3(\gamma)+\psi'(d)w'^3\delta=0,$$

where α, β, γ, δ are the circles of reference when the sphero-quartic is given by its canonical form.

CHAPTER X.

Classification of Cyclides.

185. Following the analogy of the method given in my memoir on 'Bicircular Quartics,' I shall take as the basis of classification the species of the focal quadric.

The principal varieties of quadrics are:—1°. An ellipsoid or hyperboloid. 2°. A sphere. 3°. A paraboloid. We shall find the cyclides corresponding to these varieties to have fundamental distinctions. We shall therefore devote a section to each.

SECTION I.—*Focal Quadric an Ellipsoid or Hyperboloid.*

186. Figure of cyclide. Let us denote the sphere of inversion by U, and the focal quadric by F. 1°. When the developable circumscribing U and F is imaginary, as for instance when F is an ellipsoid and U entirely within it, the cyclide evidently consists of two distinct sheets, which are inverse to each other with respect to U. One sheet is internal to U, and the other external; each sheet is a closed surface.

2°. When the developable is real, and when U does not intersect F, or else when it does intersect it in a sphero-quartic consisting of two distinct ovals, the cyclide W is made up of two closed surfaces, each of which is an anallagmatic, and divided by U into two parts. The points where W cuts U are the points of contact of the common tangent developable circumscribed to U and F.

3°. When the developable is real and the sphero-quartic of intersection of U and F

consists of one oval, W consists of one closed surface which is divided into two parts by U.

4°. When U touches F, the point of contact will be a nodal point on the cyclide, the cone of contact with the cyclide being real or imaginary according as U touches F on the concave or convex side of F (see art. 76).

Cor. If a cyclide has either a real or imaginary conic node (contracted by Professor CAYLEY into cnic-node), it arises from a real double point or a conjugate point on one of its focal sphero-quartics.

5°. When U has stationary contact with F, the point of osculation will be a biplanar node on the cyclide. In this case the cyclide will be the inverse of a non-central quadric (see art. 76).

6°. When U has double contact with F, the cyclide will be binodal.

7°. When U is inscribed in F (that is, when U touches F along a circle), the cyclide will break up into two spheres.

187. *Double Tangent Cones.*—Let us consider a cyclide whose focal quadric is F; then, taking the limiting points P, P′ of U and any tangent plane to F, the generating sphere through P, P′ will become a plane if its centre be at infinity, and the locus of the points P, P′ will evidently be a sphero-quartic, which is given as the intersection of a sphere concentric with F, and a cone whose edges are perpendicular to the tangent planes of the asymptotic cone of F, the vertex of the cone being the centre of U; this cone will be a double tangent cone. Hence we have the following theorem:—*Every cyclide has as many double tangent cones as it has focal quadrics.*

188. *The lines of intersection of a cyclide with its spheres of inversion are lines of curvature on the cyclide.*

For let us consider any point on the cuspidal edge of Σ, the developable which circumscribes U along WU; then that point is the centre of an osculating sphere of W (see art. 169). Hence WU is a line of curvature on W.

Cor. 1. The cuspidal edge of Σ is a geodesic on the surface of centres of W.

Cor. 2. The sphero-quartic reciprocal to W with respect to U^2 (that is, to $\alpha^2+\beta^2+\gamma^2+\delta^2$) is such that the focal sphero-quartic of W lying on the sphere U is a line of curvature on it.

189. *Binodal Cyclides of a Cyclide.*—We have seen (art. 33) that the cyclide W may be written in five different ways, (I.), (II.), (III.), (IV.), (V.). Now taking the first (I.), its equation is

$$(a-b)\beta^2+(a-c)\gamma^2+(a-d)\delta^2+(a-e)\varepsilon^2=0,$$

and the square of the corresponding sphere of inversion is $\beta^2+\gamma^2+\delta^2+\varepsilon^2$; and eliminating in succession each of the four letters β^2, γ^2, δ^2, ε^2, we get four binodal cyclides, each touching W along the line of curvature WU. *Hence every cyclide has in general four times as many binodals inscribed in it as it has spheres of inversion.*

190. *The imaginary circle at infinity is a flecnodal curve on the surface of centres of a cyclide.* This proposition is an extension of art. 52 in 'Bicircular Quartics.' It is proved as follows:—It is evident that the normal at any point of the imaginary circle at

infinity lies in the plane touching the cyclide along a tangent line to the circle at infinity; hence the tangent plane to the cyclide is also a tangent plane to the surface of centres.

Again, the sphere of curvature at any point P of a cyclide is the quadric through the imaginary circle at infinity and through four consecutive points at P; if P be any point on the circle at infinity, this quadric is indeterminate, and the pole of the circle at infinity is any point on the tangent plane at P. Hence any point on the tangent plane may be regarded as a point of intersection with a consecutive tangent plane; in other words, the tangent plane to the cyclide at any point along the imaginary circle at infinity is a stationary tangent plane to the surface of centres.

Cor. If the imaginary circle at infinity be a cuspidal curve on the cyclide, it will be a cuspidal curve on the surface of centres of the cyclide.

191. The points of contact of tangent lines from any point to a cyclide of the fourth degree W are the points of intersection of W with the polar cubic of the point; but this polar cubic is evidently a cubic cyclide. *Hence the tangent cone which circumscribes a cyclide and has any point for vertex reduces to the eighth degree by rejecting the square of the cone to the imaginary circle at infinity.* Or thus:—Draw any plane through the vertex of the cone; this plane will cut the cyclide in a bicircular quartic; and this quartic being of the eighth class, eight tangents can be drawn to it from the vertex of the cone.

192. *Class of Tangent Cone.*—Let V be the vertex of the tangent cone and V′ any other point, then the class of the tangent cone is plainly equal to the number of points common to W and the polar cubics of the points V, V′. Here we have three cyclides to consider, viz. W and the polar cubics. Let F, G, H be their focal quadrics; then F, G, H have eight common tangent planes; and corresponding to each common tangent plane there will be a pair of inverse points common to the cubics; therefore through the line V V′ sixteen tangent planes can be drawn to the cone; *the class of the tangent cone is therefore sixteen.*

193. The equation of the tangent cone from any point to a cyclide may be found as follows. Taking the point which is to be the vertex of the cone as origin, let the equation of the cyclide in Cartesian coordinates be written in the form

$$A(x^2+y^2+z^2)^2+4B(x^2+y^2+z^2)+6C+4D+E=0, \quad . \quad . \quad . \quad . \quad (132)$$

where A, E are constants,

$$B\equiv lx+my+nz,$$

$$D\equiv px+qy+rz,$$

$$C\equiv(a,\ b,\ c,\ f,\ g,\ h\between x,\ y,\ z)^2.$$

In polar coordinates this becomes, by putting $\rho\cos\alpha=x$, $\rho\cos\beta=y$, $\rho\cos\gamma=z$, and putting for shortness

$$B'\equiv l\cos\alpha+m\cos\beta+n\cos\gamma,$$

$$D'\equiv p\cos\alpha+q\cos\beta+r\cos\gamma,$$

$$C'\equiv(a,\ b,\ c,\ f,\ g,\ h\between\cos\alpha,\ \cos\beta,\ \cos\gamma^2),$$

$$A\rho^4+4B'\rho^3+6C'\rho^2+4D'\rho+E=0;$$

forming the discriminant of this and returning to x, y, z coordinates, we get the equation of the tangent cone to be

$$I^3-27J^2=0, \quad \ldots \quad (133)$$

where

$$I\equiv AE(x^2+y^2+z^2)^2+4BD(x^2+y^2+z^2)+3C^2, \quad \ldots \quad (134)$$

$$J\equiv ACE(x^2+y^2+z^2)^2+2BCD(x^2+y^2+z^2)-AD^2(x^2+y^2+z^2)^2-EB^2(x^2+y^2+z^2)^2-C^3. \quad (135)$$

194. From the form of the equation of the tangent cone, $I^3-27J^2=0$, it has twenty-four cuspidal edges; but from the forms of I and J we see that they have respectively, with the imaginary circle at infinity, contacts of the first and second order at each of the points where the cone C meets that circle. Hence the cuspidal edges coincide six by six with the four lines from the origin to these imaginary points; and it hence follows that, when we omit the factor $(x^2+y^2+z^2)^2$ in the equation (133), the remaining part, which represents a cone of the eighth degree, has no cuspidal edge. This equation is

$$\left.\begin{array}{l} A^3E^3\varrho^8-12A^2BDE^2\varrho^6 \\ \quad -(6AB^2D^2E+18A^2C^2E^2-54A^2CD^2E+27A^2D^4-54AB^2CE^2+27B^4E^2)\varrho^4 \\ \quad -(180AB^2C^2DE-108ABCD^3+64B^3D^3)\varrho^2 \\ \quad -(54AC^3D^2-81AC^4E+54B^2C^3E+36B^2C^2D^2)=0 \end{array}\right\}. \quad (136)$$

In this equation for shortness we have written ϱ^2 for $x^2+y^2+z^2$.

Cor. If the origin be on the cyclide $E=0$, and the tangent cone reduces to the square of the tangent plane to the cyclide at the origin and a cone of the sixth degree,

$$27A^2D^2\varrho^4+4BD(16B^2-27AC)\varrho^2-18C^2(2B^2+3AC)=0. \quad \ldots \quad (137)$$

195. The cone J is such that every edge of it is cut harmonically by the cyclide; and therefore, if any edge of it meet the cyclide in two coincident points, there must be a third point coincident; therefore, since the imaginary circle at infinity is a double line on the surface, the points where J meets it are such that every edge which passes through it is an inflectional tangent. Hence from any point can be drawn to a cyclide twelve lines, which are inflectional tangents to it at the imaginary circle at infinity; and these lines are distributed into four sets of three lines each, each triad consisting of three consecutive lines.

196. If the cyclide W has a double point, its class is diminished by two; if it has a biplanar node, its class will be diminished by three; if it has two nodes, its class will be diminished by four. The following Table contains the singularities of the tangent cones for each of these cases:—

No node.	Conic node.	Biplanar node.	Two nodes.
$m=8$,	$m=8$,	$m=8$,	$m=8$,
$n=16$,	$n=14$,	$n=13$,	$n=12$,
$\varkappa=0$,	$\varkappa=6$,	$\varkappa=9$,	$\varkappa=8$,
$\delta=20$,	$\delta=12$,	$\delta=8$,	$\delta=10$,
$\iota=24$,	$\iota=24$,	$\iota=24$,	$\iota=20$,
$\tau=80$,	$\tau=51$,	$\tau=38$,	$\tau=32$.

SECTION II.—*Focal Quadric a Sphere.*

197. When the focal quadric is a sphere, the cyclide has the imaginary circle at infinity as a cuspidal edge; on this account we shall call the surface a Cartesian cyclide.

Figure of the Surface.

1°. When U is external to F, W consists of two distinct sheets, each intersecting U in a circle. Each sheet is a closed surface.

2°. When U is internal to F, W consists of one sheet internal to U, and another sheet the inverse of the former, and therefore external to U. Each sheet is a closed surface.

3°. When U intersects F, W consists of one sheet; this intersects U in one real circle and another imaginary circle. The sheet is a closed surface.

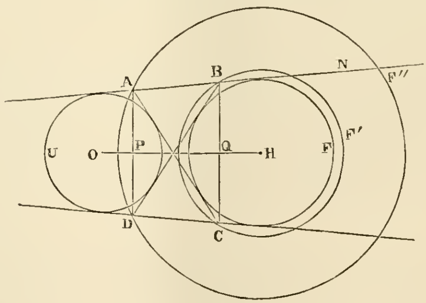
4°. When U touches F internally, W has a conic node, the tangent cone to W at the node being a real cone of revolution.

5°. When U touches externally, W has a conic node at which the tangent cone is imaginary.

6°. When U reduces to a point, W is the pedal of a sphere, and is therefore the inverse of a quadric of revolution from the focus.

198. In the annexed diagram, which is supposed to be a plane section through the centres of U and F, let B, C and A, D be the opposite pairs of the intersections of the tangent cones circumscribed about U and F made by the plane of the section, then the spheres F′, F″ concentric with F, and passing respectively through B, C and A, D, are

Fig. 5.



focal spheres (see art. 34) of the Cartesian cyclide; for the developable circumscribed about U and F reduces in this case to two cones of revolution, and the nodal lines of the geometric system composed of the two cones are two circles which are intersected by the plane of the section in B, C, A, D and the vertices of the cones. Hence the line O H passing through the two vertices must be regarded as a limiting case of an hyperboloid confocal with the spheres F, F′, F″ (see art. 106). Hence the focal quadrics of a Cartesian cyclide are three concentric spheres and a straight line through their centre.

199. Let OH intersect AD and BC in P and Q, then P and Q are the limiting points of U and F; and if we denote the radii of U and F by r and R, we have $\mathrm{OP}.\mathrm{OQ}=r^2$, and $\mathrm{HP}.\mathrm{HQ}=\mathrm{R}^2$; but since the lines AO, AH are evidently the bisectors at A of the supplemental angles made by the tangents, they are at right angles to each other, and in like maner the lines OB, BH are at right angles to each other. Hence the points Q and O are inverse points with respect to F′, and P and O with respect to F″. Again, it is easy to see that Q and O are inverse points with respect to the imaginary sphere U′ whose centre is P, and which cuts U orthogonally, and P and O with respect to U″ whose centre is Q, and which cuts U orthogonally. Hence the limiting points of U′ and F′ are the centres of U″ and U; the limiting points of U″ and F″ are the centres U and U′; so that the limiting points of any U and its corresponding F are the centres of the two remaining U's or spheres of inversion.

200. The centres of inversion of a Cartesian cyclide are foci of the surface.

Demonstration. Let the equations of U and F be

$$x^2+y^2+z^2=r^2, \text{ and } (x+a)^2+y^2+z^2=\mathrm{R}^2,$$

then the perpendicular OT let fall from O, the centre of U on a tangent plane to F, is evidently equal to $\mathrm{R}-a\cos\theta$, where θ is the angle which the perpendicular makes with the axis of x; and if P, P′ be points on OT such that $\mathrm{OT}^2-\mathrm{TP}^2=\mathrm{OT}^2-\mathrm{TP}'^2=r^2$, then P, P′ are points on the Cartesian cyclide; and denoting OP by ρ, we have

$$2(\mathrm{R}-a\cos\theta)\rho=r^2+\rho^2,$$

or

$$2\mathrm{R}\rho=x^2+y^2+z^2+2ax+r^2;$$

that is,

$$4\mathrm{R}^2(x^2+y^2+z^2)=(x^2+y^2+z^2+2ax+r^2). \quad . \quad . \quad . \quad . \quad . \quad (138)$$

Hence $x^2+y^2+z^2=0$ is an imaginary cone circumscribed to the cyclide. Hence the centre of U is a focus of the surface.

201. The equation (138) may evidently be written in the form $\mathrm{S}^2=b^3\mathrm{L}$, where S is a sphere and L a plane, showing that the imaginary circle at infinity is a cuspidal edge on the surface.

The equation of the sphere S is found to be

$$x^2+y^2+z^2+2ax+r^2-2\mathrm{R}^2=0;$$

and this is concentric with the focal spheres F, F′, F″; but the centre of S is a triple focus, as appears from the equation $\mathrm{S}^2=b^3\mathrm{L}$.

Hence the common centre of the three focal spheres F, F′, F″ *is the triple focus of the Cartesian cyclide. The Cartesian cyclide has a tangent plane which touches it along a circle; the plane is* L; *and the circle of contact is the circle of intersection of* S *and the plane* L.

202. Since being given the sphere of inversion U and the focal sphere F the Cartesian cyclide is determined, we see that a Cartesian cyclide is determined by eight constants. The same thing appears from the equation $\mathrm{S}^2=b^3\mathrm{L}$. From this equation also

we see that if a Cartesian cyclide be intersected by any plane the curve of intersection will be a Cartesian oval; for the equation will be of the form $S^2=b^3l$, where S and l denote the circle and line in which the sphere S and the plane L are intersected by the plane.

203. From the equation (138) we see that the Cartesian cyclide is the envelope of the variable sphere

$$x^2+y^2+z^2+\mu(x^2+y^2+z^2+2ax+r^2)+\mu^2R^2;$$

and if we form the discriminant of this, we get

$$(1+\mu)^2\{(1+\mu)(\mu r^2+\mu^2R^2)-a^2\mu^2\}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (139)$$

Now the factor $(1+\mu)^2=0$ gives $\mu=-1$; and for this value of μ the variable sphere becomes a plane, namely the tangent plane which touches the cyclide along a circle; the remaining factor,

$$(1+\mu)(\mu r^2+\mu^2R^2)-a^2\mu^2=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (140)$$

gives three values of μ, for each of which the variable sphere becomes an imaginary cone (that is, a point sphere), showing that there are three collinear single foci along the axis of the cyclide. The value $\mu=0$ shows that the origin is a focus, which we knew before; and the values giving the other foci are the roots of the quadratic

$$\mu^2R^2+\mu(R^2+r^2-a^2)+r^2=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (141)$$

204. Since the equation

$$x^2+y^2+z^2+\mu(x^2+y^2+z^2+2ax+r^2)+\mu^2R^2=0$$

is that of a sphere into whose equation an arbitrary constant enters in the second degree, its inverse with respect to any point will be a sphere into whose equation an arbitrary constant enters in the second degree; that is, *the inverse of a Cartesian cyclide will be a cyclide generated as the envelope of a variable sphere whose centre moves along a plane conic. It will therefore be a binodal cyclide.*

This also appears from the fact that the inverse of a focus is a focus; and since the Cartesian cyclide has three collinear single foci, the inverse surface will have four concyclic single foci, namely the inverses of the three collinear foci and the centre of inversion.

205. If we differentiate the equation (138) of the generating sphere with respect to μ, we get

$$x^2+y^2+z^2+2ax+r^2+2\mu R^2=0;$$

and if from $(1+\mu)$ times this result we subtract the equation (138), we get

$$2ax+r^2+(2\mu+\mu^2)R^2=0.$$

Hence the Cartesian cyclide is generated as the locus of the curve of intersection of the sphere

$$x^2+y^2+z^2+2ax+r^2+2\mu R^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (142)$$

with the plane

$$2ax+r^2+(2\mu+\mu^2)R^2=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (143)$$

From this it follows that a Cartesian cyclide is a surface of revolution—a fact which

we knew otherwise, it being the surface generated by the revolution of a Cartesian oval about the axis passing through the three collinear single foci.

206. In order to find the equation of the cone whose vertex is any point x', y', z', and which stands on the circle of intersection of the sphere (142) and the plane (143), let us suppose x'', y'', z'' to be any point on a radius vector from x', y', z' to any point of the circle; then, if the circle divides the distance between these points in the ratio $l:m$, we must substitute, by JOACHIMSTAL's method, $\frac{lx'+mx''}{l+m}$ $\frac{ly'+my''}{l+m}$ $\frac{lz'+mz''}{l+m}$ for x, y, z in (142) and (143); the results will be of the form

$$l^2S'+2lmP+m^2S''=0, \text{ and } lL'+mL''=0;$$

hence by eliminating $l:m$ and suppressing the double accents, we get the required equation after restoring the values of S', P, &c.:—

$$\left.\begin{array}{l}(2ax+r^2+(2\mu+\mu^2)R^2)^2(x'^2+y'^2+z'^2+2ax'+r^2+2\mu R^2)\\ -2(2ax+r^2+(2\mu+\mu^2)R^2)(2ax'+r^2+(2\mu+\mu^2)R^2)(xx'+yy'+zz'+ax+ax'+r^2+2\mu R^2)\\ +(2ax'+r^2+(2\mu+\mu^2)R^2)(x^2+y^2+z^2+2ax+r^2+2\mu R^2)=0.\end{array}\right\}(144)$$

207. Since the equation (144) involves the undetermined μ in the fourth degree, its discriminant with respect to μ will involve x, y, z in the twelfth degree, and this discriminant will be the equation of the tangent cone; but this will contain as a factor the cube of the imaginary cone from (x', y', z') to the imaginary circle at infinity (see SALMON's 'Geometry of Three Dimensions,' art. 521). Hence the reduced degree is six; and it can be shown, as in art. 94, that the reduced cone has no cuspidal edges.

We can show otherwise that the reduced degree is six; for any section of the cyclide made by a plane through the vertex of the cone is a Cartesian oval, and the class of a Cartesian oval is six; the degree of the cone therefore is six.

208. *Class of Tangent Cone.*—Let us take the equation $S^2=b^3L$ and find the polar cubic of the point x', y', z'. This will be of the form

$$2S\left(x'\frac{d}{dx}+y'\frac{d}{dy}+z'\frac{d}{dz}\right)S=b^3\left(x'\frac{d}{dx}+y'\frac{d}{dy}+z'\frac{d}{dz}\right)L;$$

and eliminating b^3 between this and the equation $S^2=b^3L$, we get

$$S\left(x'\frac{d}{dx}+y'\frac{d}{dy}+z'\frac{d}{dz}\right)L=2L\left(x'\frac{d}{dx}+y'\frac{d}{dy}+z'\frac{d}{dz}\right)S;$$

and since the operation $x'\frac{d}{dx}+y'\frac{d}{dy}+z'\frac{d}{dz}$ performed upon L reduces it to a constant, and performed upon S reduces it to a plane, this equation represents a quadric. Hence it is easy to see that the points of contact of tangent planes drawn through a line to the tangent cone are the intersections of three surfaces of the degrees 3, 2, 2; hence the class of the tangent cone is 12; and we have shown that its degree is six, and that it has no cuspidal edges. Hence all the singularities are determined.

SECTION III.—*Focal Quadric a Paraboloid.*

209. When the focal quadric is a paraboloid, the cyclide becomes a cubic surface passing through the imaginary circle at infinity. The varieties of this surface correspond to those of quartic cyclides, and may be briefly enumerated as follows:—

1°. When the developable circumscribed to U and F is imaginary, the surface consists of two sheets, one of which is a closed surface passing through the centre of U and altogether within U. The other sheet, which is its inverse of the first, is an open sheet, extending to infinity, which it intersects in a right line. The case considered here would occur if F were an elliptic paraboloid, and U in the concavity of it without meeting it.

2°. When the developable is real the surface consists, as in 1°, of two sheets, one of which is a closed surface and passes through the centre of U, the other sheet is infinite. Each sheet intersects U; and the part of each sheet internal to U is the inverse of the external part.

3°. When U intersects F in a single oval, the cyclide consists of one infinite sheet passing through the centre of U.

4°. When U touches F, we have to consider separately the cases where F is an elliptic paraboloid and where a hyperbolic paraboloid.

If F be an elliptic paraboloid, and U touch it on the convex side, the cyclide has a conic node, whose tangent cone is imaginary.

If U touch F on the concave side, the cyclide has a node whose tangent cone is real; and if U touch F at an umbilic, the tangent cone to the node is one of revolution: lastly, if U osculate F at an umbilic, the tangent cone becomes a plane; that is, each sheet of the cone opens out into a plane, and the node is a biplanar node whose planes coincide. In the case where F is a hyperbolic paraboloid, when U touches it we have a conic node whose tangent cone is always real, but which becomes a pair of planes if U osculate F.

5°. When U has double contact with F, the cyclide will be binodal.

210. In the examination we have given in this and the previous sections of this chapter, we have seen that a cyclide of any class can have but three species of node (namely, the conic node, the biplanar node, and the uniplanar node), and that these correspond respectively to ordinary contact of the sphere of inversion and the focal quadric, oscular contact, and oscular contact at an unibilic. From this it follows that a cyclide can at most have but two real nodes; and if it has two, they must be conic nodes; for if it had a conic node and a biplanar node, U should touch and osculate F at the same time; that is, U should intersect F in a quartic curve having a double point and a cusp, and the plane through the cuspidal tangent and the double point would intersect a quartic curve in five points, which it cannot do; and of course a cyclide cannot, for a like reason, have two biplanar nodes.

By a different mode of reasoning it may be shown that a cubic cyclide cannot have three real nodes; for if it had, the plane through the nodes must intersect the cubic cyclide

in three right lines, and then, by inversion, from any point in the plane we should have the absurdity of a quartic cyclide being intersected by a plane in three circles.

211. *Parallel Tangent Planes.*—Let us consider a cubic cyclide whose sphere of inversion is U and focal paraboloid F. If P, P′ be the limiting points of U and the tangent plane to F at infinity, then of the two points P, P′ one must be at the centre of U and the other at infinity; and it is plain that the generating sphere which touches the cyclide at P, P′ must break up into two planes, namely the tangent planes to the cyclide at P, P′. The tangent plane at the centre of U must evidently be parallel to the principal plane of the paraboloid which does not intersect it in a parabola; and since the cyclide has five centres of inversion, *we see that every cubic cyclide has five parallel tangent planes, and these are the five tangent planes which can be drawn to a cubic cyclide from the line at infinity on the cyclide.* Hence we infer the following theorem:—*The five tangent planes to a cubic cyclide from the line at infinity on the cyclide have the five centres of inversion as points of contact.*

212. The property of the last article may be shown otherwise. Thus, consider any quartic cyclide; then at any point Q five generating spheres touch, namely one belonging to each of the five systems of generating spheres. Now, since a generating sphere intersects a cyclide in two circles, if we invert the quartic cyclide from the point Q we get a system of five parallel planes, each intersecting the cubic cyclide into which the quartic inverts in two lines, and therefore having the points of intersection of these lines as points of contact with the cyclide.

213. The section of a cubic cyclide made by a plane passing through any line on the cyclide except the line at infinity must consist of the line and a circle; for it must consist of a line and a conic, and by inverting from any point the line and conic must invert into a bicircular quartic; hence the conic must be a circle.

This reasoning will not apply in the case of a section made by a plane through the line at infinity; for when we invert the line at infinity it becomes a point, which accounts for the double point which results when a conic is inverted; so that when we say the inverse of a plane conic is a bicircular quartic, this includes the inverse of the line at infinity together with that of the conic.

214. Since the five centres of inversion of a cubic cyclide form a pentahedron, such that, taking any four of them forming a tetrahedron, the perpendiculars of that tetrahedron are concurrent and intersect in the fifth point, we see without difficulty that the feet of the perpendiculars of the pentahedron are points on the cubic cyclide, and we may easily infer the following theorem:—

Being given eight homospheric points (say, eight points on the sphere U*), three cubic cyclides can be described having these eight points as foci. These cyclides intersect two by two orthogonally, they have the same five centres of inversion, and each passes through the feet of the perpendiculars of the pentahedron.*

215. In order to find the equation of the tangent cone to a cubic cyclide from any given point, let the given point be taken as the origin of Cartesian coordinates, and

we shall have

$$W \equiv A(x^2+y^2+z^2)+3B+3C+D, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (145)$$

where

$$A \equiv lx+my+nz, \qquad B \equiv (a, b, c, f, g, h \between x, y, z)^2,$$

$$C \equiv px+qy+rz, \qquad D \equiv \text{constant};$$

then, by the method of art. 193, we find the tangent cone

$$G^2-4H^3 \equiv \text{discriminant} \times A^6, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (146)$$

where

$$G \equiv A^2D(x^2+y^2+z^2)^2+2B^3+3ABC(x^2+y^2+z^2)=0,$$

$$H \equiv B^2-AC(x^2+y^2+z^2).$$

Hence the tangent cone is

$$A^2D^2(x^2+y^2+z^2)^2+(4AC^3-6ABCD)(x^2+y^2+z^2)+4DB^3-3B^2C^2=0. \quad . \quad (147)$$

Cor. The plane A is parallel to the tangent planes to the cyclide at its centres of inversion.

216. The cone G possesses the property that any edge of it meets the cyclide in three points, whose distances from the vertex are in arithmetical progression. Now, if we invert a cubic cyclide from the vertex of the cone, we get a quartic cyclide; and since the cone G meets the cubic cyclide in points whose distances from the vertex are in arithmetical progression, it will meet the quartic cyclide in points whose distances are in harmonical progression. Hence the cone G is identical with the cone J of article 193, when the vertex of J is on the surface.

217. Since a cubic cyclide is determined when U and F are given, and U is determined by four and F by eight conditions, we see that a cubic cyclide is determined by $4+8=12$ conditions. Hence it follows that every cubic cyclide can be written in the form

$$A\alpha=B\beta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (148)$$

where A and B are planes, and α and β spheres; and in this form it is evident that the intersection of the radical plane of the spheres α, β with the two planes A, B is a centre of inversion of the cyclide. From the equation (148), being the result of eliminating k between the equations $A-kB=0$ and $k\alpha-\beta=0$, we infer that *if we have a system of planes passing through the same line, and a homographic system of spheres passing through the same circle, the locus of the circle of intersection of a sphere and its corresponding plane is a cubic cyclide.*

CHAPTER XI.

Classification of Sphero-quartics.

218. We have seen that if $W \equiv a\alpha^2+b\beta^2+c\gamma^2+d\delta^2$, and $U^2 \equiv \alpha^2+\beta^2+\gamma^2+\delta^2$, the sphero-quartic WU is also the curve of intersection of the quadric $V \equiv ax^2+by^2+cz^2+dw^2$ and the sphere $U \equiv x^2+y^2+z^2+w^2$, the tetrahedron of reference being the one formed by the four planes of intersection of U with the four orthogonal spheres $\alpha, \beta, \gamma, \delta$.

Hence in this section we shall discuss the curve WU by regarding it as the intersection of V and U.

For the purpose of classification I shall, following CAYLEY and SALMON, consider the curve (UV) as made up of points; then the points of UV will be the points of the system, the line joining two consecutive points will be a line of the system, and the plane of two consecutive lines will be a plane of the system. If a plane of the system contains four consecutive points it will be a stationary plane; and reciprocally, if four consecutive planes of the system intersect in a point of the system, it will be a stationary point. Again, if a line join two non-consecutive points it will be a line through two points; reciprocally, if a line be the intersection of two non-consecutive planes, it will be a line in two planes; finally, if two non-consecutive lines intersect, their point of intersection will be a point on two lines, and their plane a plane through two lines. For the purpose of denoting these singularities the following notation will be used.

Thus we shall denote by

r, the number of lines of the system which meet an arbitrary line.		
m,	,,	points of the system which lie in any plane.
α,	,,	stationary planes of the system.
x,	,,	points on two lines which lie in a given plane.
g,	,,	lines in two planes which lie in a given plane.

The reciprocals of m, α, x, g will be denoted by the letters respectively consecutive to them, namely, n, β, y, h.

m is called the degree of the system.			
n	,,	class	,,
r	,,	rank	,,

219. The complete surface formed by the lines of the system (UV) is the developable Δ, whose properties we have discussed in Chapter VIII., the curve (UV) being the cuspidal edge. Again, the developable Σ of Chapter VIII. formed by the tangent planes to U along the curve UV is the reciprocal of Δ, and the point, lines, and planes of Δ are respectively the reciprocals of the planes, lines, and points of Σ, with respect to U; so that when we have the characteristics of Δ, by reciprocation we shall have the characteristics of Σ.

220. Let us consider the cone whose vertex is at any point and which stands on the curve (UV). If, for the sake of distinction, we denote by Greek letters the characteristics of a plane curve (that is, if μ, ν, δ, τ, $\varkappa$, ι denote the degree, class, double points, double tangents, cusps, points of inflection of the curve), then, for a cone standing on that plane curve, it is evident that the same letters will denote the degree, class, double edges, double tangent planes, cuspidal edges, and stationary tangent planes; then for the cone on (UV) $\mu=m$, $\nu=r$, $\delta=h$, $\tau=y$, $\varkappa=\beta$, $\iota=n$. (See SALMON's 'Geometry of Three Dimensions,' art. 321.)

Hence we get, by PLÜCKER's equations,

$$r=m(m-1)-2h-3\beta, \qquad m=r(r-1)-2y-3n,$$
$$n=3m(m-2)-6h-8\beta, \qquad \beta=3r(r-2)-6y-8n,$$
$$(n-\beta)=3(r-m), \qquad 2(y-h)=(r-m)(r+m-9).$$

221. Again, let us consider a plane section of Δ. Professor CAYLEY has shown that for such a section

$$\mu=r,\ \nu=n,\ \delta=x,\ \tau=g,\ \kappa=m,\ \iota=\alpha.$$

Hence, from PLÜCKER's equations, we get

$$n=r(r-1)-2x-3m, \qquad r=n(n-1)-2g-3\alpha,$$
$$\alpha=3r(r-2)-6x-8m, \qquad m=3n(n-2)-6g-8\alpha.$$

Whence also Dr. SALMON gets

$$(m-\alpha)=3(r-n), \qquad 2(x-g)=(r-n)(r+n-9).$$

It is plain this system might be got from the former by considering the cone whose vertex is any point and which stands on the cuspidal edge of Σ, and then reciprocating.

If we combine the equations of this article with those of the last, we get

$$(\alpha-\beta)=2(n-m), \quad x-y=(n-m), \quad 2(g-h)=(n-m)(m+n-7),$$

since PLÜCKER's equations enable us, being given any three singularities of a plane curve, to determine all the rest. The equations of this and the preceding article enable us, being given any three singularities of a twisted curve, to determine all the rest. In a succeeding article I shall point out how they may be employed to determine the singularities of the evolute of a plane curve when three of the singularities of the original curve are given.

222. *Eight lines of* Δ *meet an arbitrary line.*

Demonstration. Let (LM) be the arbitrary line and P a point in (LM) where one of the lines of Δ meets it; then it is plain, if P be the common vertex of two cones tangential to U and V respectively, one of the four common edges of the two cones will be a line of Δ, and also that the intersection of the polar planes of P with respect to U and V will pass through the curve UV. Now the intersection of the polar planes of U and V with respect to all the points of LM is a quadric. For let x', y', z', w', x'', y'', z'', w'' be any two fixed points on (LM), and U', U'', V', V'' be their polar planes with respect to U and V; then the coordinates of any other point on (LM) will be $lx'+mx''$, $ly'+my''$, $lz'+mz''$, $lw'+mw''$, and therefore the polar planes will be $l\text{U}'+m\text{U}''$, $l\text{V}'+m\text{V}''$, and, eliminating linearly, the locus required is $\text{U}'\text{V}''-\text{U}''\text{V}'=0$, a quadric which intersects the curve UV in eight points. Hence the proposition is proved.

223. *The eight planes determined by the line* LM *with the eight lines of* Δ *which meet it are tangent planes to the reciprocal of the quadric* $\text{U}'\text{V}''-\text{U}''\text{V}'$ *with respect to* U. For since eight lines of Δ meet LM, eight lines of Σ meet the polar line of LM with

respect to U, and the eight planes are the reciprocals of the eight points of meeting; but these eight points of meeting lie on a generator of $U'V''-U''V$. Hence the proposition is proved.

Cor. 1. The line LM is a generator of the reciprocal of $U'V''-U''V'$.

Cor. 2. The quadric $U'V'-U''V'$ is the hyperboloid generated by the polar lines of LM with respect to all the quadrics of the pencil $U+kV$. In fact the polar lines form one system of generators, and the intersection of the polar planes of art. 222 the other system of generators.

Cor. 3. The eight planes of art. 222 are homographic with the eight points in which the corresponding lines of Δ meet LM.

224. From art. 161 it is evident that Σ has sixteen stationary points. These are the points of contact with N, N', N'', N''' of the common tangents of J and N, J', N'; J'', N''; J''', N'''. Hence it follows that Δ has sixteen stationary planes. Hence we have three of the characteristics of Δ; for m evidently is equal to 4, and $r=8$ from art. 222. Therefore we have

$$m=4,\ \alpha=16,\ r=8.$$

Hence by CAYLEY'S equation we get

$$n=12,\ \beta=0,\ x=16,\ y=8,\ g=38,\ h=2.$$

225. We can now show the connexion which exists between the singularities determined in the last article and those of bicircular quartics. Let us suppose a cone whose vertex is any point on U, and which stands on UV; then for such a cone we have the singularities (see art. 220)

$$\mu=4,\ \nu=8,\ \delta=2,\ \tau=8,\ \varkappa=0,\ \iota=12;$$

but if we invert the sphere U into a plane, taking the vertex of the cone as centre of inversion, UV will be inverted into a bicircular quartic, which will be the curve of intersection of the cone with the plane into which U inverts, and therefore having the same singularities as the cone. The numbers here determined are therefore the singularities of a bicircular quartic (see 'Bicircular Quartics,' art. 46).

226. Again, to show the connexion with the evolute of a bicircular, let us consider a plane section of Δ; the characteristics are (see art. 221)

$$\mu=8,\ \nu=12,\ \delta=16,\ \tau=38,\ \varkappa=4,\ \iota=16.$$

Now the cone whose vertex is the pole with respect to U of the plane of sections, and which stands on the cuspidal edge of Σ, will be the reciprocal of the section; and if the plane of section be a tangent plane to U, its pole will be a point on U; therefore the singularities of this cone will be

$$\mu=12,\ \nu=8,\ \delta=38,\ \tau=16,\ \varkappa=16,\ \iota=4;$$

but when the sphere is inverted into a plane as in the last article, it has been shown in art. 92 that the cone here considered, viz. the one standing on the cuspidal edge of Σ, intersects the plane into which the sphere inverts in the evolute of the bicircular into

which UV inverts; consequently the evolute has the same singularities as the cone (see art. 51, 'Bicircular Quartics').

227. If U and V touch, the singularities of Δ are (see SALMON's 'Geometry of Three Dimensions,' art. 342)

$$m=4,\quad g=6,\quad \beta=0,$$
$$n=6,\quad h=3,\quad x=6,$$
$$r=6,\quad \alpha=4,\quad y=4.$$

Hence, by the method of the last two articles, we get for the bicircular and its evolute the following singularities:—

Bicircular, $\mu=4,\ \nu=6,\ \delta=3,\ \tau=4,\ \kappa=0,\ \iota=6.$

Evolute, $\mu=6,\ \nu=6,\ \delta=6,\ \tau=6,\ \kappa=4,\ \iota=4.$

This bicircular is the inverse of an ellipse or hyperbola; and the characteristics of the bicircular are the reciprocals of the characteristics of the evolute of an ellipse or hyperbola.

228. If U and V osculate, we have (see SALMON's 'Geometry of Three Dimensions,' art. 342) the singularities

$$m=4,\quad g=2,\quad \beta=1,$$
$$n=4,\quad h=2,\quad x=2,$$
$$r=5,\quad \alpha=1,\quad y=2.$$

Hence for the bicircular and its evolute we get

Bicircular, $\mu=4,\ \nu=5,\ \delta=2,\ \tau=2,\ \kappa=1,\ \iota=4.$

Evolute, $\mu=4,\ \nu=5,\ \delta=2,\ \tau=2,\ \kappa=1,\ \iota=4.$

This bicircular is the inverse of a parabola; and we see that it has the same characteristics as its evolute (see foot-note, art. 70, 'Bicircular Quartics').

229. By considering special sections of Δ and Σ we get, as in the preceding articles, the singularities of bicircular quartics, Cartesian ovals, circular cubics, and the evolutes of these respective species of curves. The reader who has followed out the method of reasoning in recent articles can easily account for the result in each case. In order to save space, I shall give only the particular cone whose intersection with the inverse of the sphere U gives the bicircular and the evolute. The numbers for the several cones are taken from SALMON's 'Geometry of Three Dimensions,' art. 324 (see also Cambridge and Dublin Mathematical Journal, vol. v. p. (23)–(46)).

I. When U and V do not touch.

1°. Cone whose vertex is a point of the system Δ:

$$\left.\begin{matrix}\mu=3, & \kappa=0,\\ \nu=6, & \tau=0,\\ \iota=9, & \delta=0.\end{matrix}\right\}\ \text{A circular cubic of the sixth class.}$$

2°. Cone whose vertex is a point on two lines of Δ:

$$\left.\begin{array}{ll}\mu=4, & \kappa=2,\\ \nu=6, & \tau=1,\\ \iota=9, & \delta=0.\end{array}\right\}\text{ A Cartesian oval, sixth class.}$$

3°. Cone whose vertex is a point on a line of Σ and which stands on the cuspidal edge of Σ:

$$\left.\begin{array}{ll}\mu=12, & \varkappa=37,\\ \nu=\ 7, & \tau=\ 2,\\ \iota=\ 2, & \delta=37.\end{array}\right\}\text{ Evolute of circular cubic of sixth class.}$$

4°. Cone whose vertex is a point on two lines of Σ and which stands on the cuspidal edge of Σ:

$$\left.\begin{array}{ll}\mu=12, & \varkappa=18,\\ \nu=\ 6, & \tau=\ 9,\\ \iota=\ 0, & \delta=36.\end{array}\right\}\text{ Evolute of Cartesian oval of sixth class.}$$

II. When U and V touch.

1°. Cone whose vertex is a point of Δ:

$$\left.\begin{array}{ll}\mu=3, & \varkappa=0,\\ \nu=4, & \tau=0,\\ \iota=3, & \delta=1.\end{array}\right\}\text{ A circular cubic of fourth class.}$$

2°. Cone whose vertex is a point on two lines of Δ:

$$\left.\begin{array}{ll}\mu=4, & \varkappa=2,\\ \nu=4, & \tau=1,\\ \iota=2, & \delta=1.\end{array}\right\}\text{ A Cartesian oval of fourth class.}$$

3°. Cone whose vertex is a point on a line of Σ and which stands on the cuspidal edge of Σ:

$$\left.\begin{array}{ll}\mu=6, & \varkappa=5,\\ \nu=5, & \tau=4,\\ \iota=2, & \delta=5.\end{array}\right\}\text{ Evolute of circular cubic of fourth class.}$$

4°. Cone whose vertex is a point on two lines of Σ and which stands on the cuspidal edge of Σ:

$$\left.\begin{array}{ll}\mu=6, & \varkappa=6,\\ \nu=4, & \tau=3,\\ \iota=0, & \delta=4.\end{array}\right\}\text{ Evolute of Cartesian oval of fourth class.}$$

III. When U and V osculate.

1°. Cone whose vertex is a point of Δ:

$$\left.\begin{array}{ll}\mu=3, & \varkappa=1, \\ \nu=3, & \tau=0, \\ \iota=1, & \delta=0.\end{array}\right\} \text{Circular cubic of third class.}$$

2°. Cone whose vertex is a point on two lines of Δ:

$$\left.\begin{array}{ll}\mu=4, & \varkappa=3, \\ \nu=3, & \tau=1, \\ \iota=0, & \delta=0.\end{array}\right\} \text{A Cartesian oval of third class—that is, a cardioide.}$$

3°. Cone whose vertex is a point on a line of Σ and which stands on the cuspidal edge of Σ:

$$\left.\begin{array}{ll}\mu=4, & \varkappa=2, \\ \nu=4, & \tau=1, \\ \iota=2, & \delta=1.\end{array}\right\} \text{Evolute of circular cubic of third class.}$$

4°. Cone whose vertex is a point on two lines of Σ and which stands on the cuspidal edge of Σ:

$$\left.\begin{array}{ll}\mu=4, & \varkappa=3, \\ \nu=3, & \tau=1, \\ \iota=0, & \delta=0.\end{array}\right\} \text{Evolute of a cardioide.}$$

230. If a plane curve whose degree is N be inverted from any point out of the plane of the curve, it will invert into a twisted curve, whose characteristics are easily found.

1°. Let us suppose that the plane curve does not pass through the circular points at infinity. Then, since the curve passes through N points at infinity, the inverse curve will have this order of multiplicity at the origin of inversion; and since the plane of the curve will invert into a sphere, the inverse curve will be the intersection of two surfaces of the degrees N and 2 respectively, having a multiple contact equivalent to $\frac{N(N-1)}{2}$ points of ordinary contact. Hence we have $m=2N$, $\beta=0$, $2h=3N^2-3N$ (see SALMON, p. 273). Hence, by CAYLEY'S equations,

$$r=N^2+N, \qquad n=3N^2-3N, \qquad 2y=N^4+2N^3-9N^2+6N,$$
$$\alpha=6N^2-10N, \quad 2x=N^4+2N^3-3N^2-4N, \quad 2g=9N^4-18N^3-13N^2+32N.$$

231. Next, let us suppose that the plane curve passes k times through each of the circular points at infinity; then it is easy to see that the inverse curve will be the intersection of a sphere and a surface of the degree $N-k$. Hence we have in this case for determining the singularities, $m=2(N-k)$, $\beta=0$, and $2k=3N^2-8Nk+6k^2-3N+4k$.

Hence by CAYLEY'S equations we get the following results for the other singularities:

$$r = N^2 - 2k^2 + N - 2k,$$
$$n = 3N^2 - 6k^2 - 3N,$$
$$\alpha = 6N^2 - 12k^2 - 10N + 4k,$$
$$2y = N^4 + 2N^3 - 9N^2 + 6N - 4(N^2+N-1)k - 4(N^2+N-6)k^2 + 8k^3 + 4k^4,$$
$$2x = N^4 + 2N^3 - 3N^2 - 4N - 4(N^2+N-2)k - 4(N^2+N-3)k^2 + 8k^3 + 4k^4,$$
$$2g = 9N^4 - 18N^3 - 13N^2 + 32N - 10k - 4(9N^2 - 9N - 11)k^2 + 36k^4.$$

We can easily verify these results in the case of $N=4$ and $k=2$, which is that of a bicircular quartic; they give the results previously obtained (see art. 224).

232. Let us now find the singularities of the cone whose vertex is the point we invert from, and which stands on the inverse curve. The vertex of the cone is a multiple point on the curve, the degree of multiplicity being of the order $N-2k$; but since the twisted curve is of the degree $2N-2k$, and the multiplicity of the vertex is $N-2k$, it follows that the degree of the cone is N, $\therefore \mu = N$. Again, the class of the cone is the same as the number of tangent planes which pass through an arbitrary line through the vertex;

$$\therefore \nu = r - 2(N-2k);$$

and using the value of r in the last article we get

$$\nu = N^2 - N - 2k(k-1), \text{ and } k = \beta = 0.$$

Hence by PLÜCKER'S equations the other singularities of the cone are determined. It is evident we could get all these results at once, since evidently the singularities of the cone are the same as those of the original plane curve; but getting them as done here verifies the equations of the last article.

233. Let us find the singularities of the section of the developable circumscribed to the sphere into which the plane inverts along the inverse curve made by the tangent plane to the sphere at the origin of inversion. The characteristics of the developable considered here are got from those of art. 231, by leaving r unaltered and by changing m, α, g, x into n, β, h, y, and *vice versâ*; and the plane in question will be a plane of multiple contact of the degree $N-2k$; that is, it will touch the developable along $N-2k$ lines, and will intersect it besides in a curve of the degree

$$r - 2(N-2k) = N^2 - N - 2k(k-1).$$

We have therefore

$$\mu = N(N-1) - 2k(k-1).$$

The class of the curve is determined by the number of planes of the system which can be drawn through any point of the section; and since in this case $N-2k$ planes coincide with the plane of the section itself, the number of remaining planes $\nu = N$, and we have $\iota = \alpha = 0$, and by PLÜCKER'S equation the remaining characteristics can be found. These results are the reciprocals of those found in the last article, as they evidently ought to be.

234. The most important problem in this inquiry is to find the singularities of the

cone whose vertex is the origin of inversion, and which stands on the cuspidal edge of the developable formed by the tangent planes of the sphere along the inverse curve—that is, the sphere into which the plane inverts. These singularities will be those of the evolute of the original plane curve.

The singularities of the developable will be got by the changing of letters as in the last article. Since the origin is a multiple point of the degree $N-2k$, therefore the class of the cone will be $r-(N-2k)=N^2-2k^2$, since it is evident that, in finding the number of lines of the system which meets an arbitrary line through the vertex of the cone, we must subtract from the rank of the system the number denoting the multiplicity of the vertex.

Since any arbitrary plane meets the cuspidal edge in a number of points equal to the degree of the system (that is, $3N^2-6k^2-3N$), it is evident the degree of the cone is equal to this number diminished by $N-2k$; therefore the degree of the cone is

$$3N^2-6k^2-4N+2k.$$

Again, the cuspidal edges of the cone will be equal to the number of stationary points of the system—that is, equal to

$$6N^2-12k^2-10N+4k.$$

In order to find the corresponding singularities for the evolute, we must plainly add $N-2k$ to the last two singularities; for, $N-2k$ branches of the inverse curve passing through the origin of inversion, each branch will add one to the number of cusps, and one to the degree, and we shall have for the evolute of a curve of degree N which passes k times through each of the circular points at infinity, but which has no finite double point or cusp, the following singularities,

$$\begin{aligned} \text{Class} &= N^2-\ 2k^2, \\ \text{Degree} &= 3N^2-\ 6k^2-3N, \\ \text{Cusps} &= 6N^2-12k^2-9N+2k; \end{aligned}$$

and by PLÜCKER's equations the other singularities can be determined.

By putting $N=4$ and $k=2$, we find class, degree, and cusps of a bicircular quartic to be 8, 12, 16, and our formula is verified for the bicircular.

Again, putting $N=3$ and $k=1$, we find the numbers for the circular cubic to be 7, 12, 17, which we know otherwise to be the characteristics for the evolute of a circular cubic. If we put $k=0$ in the above formulæ, the numbers coincide with those in SALMON's higher curves.

The foregoing numbers are to be modified when the curve of the Nth degree has cusps at the circular points at infinity. In that case for each cusp at a circular point at infinity the class of the evolute will be diminished by unity, and the number of its cusps increased by unity, the degree remaining the same. If the original curve had finite double points or cusps, the surfaces, viz. the sphere and the surface of the Nth degree, will have ordinary contact for each double point on the curve, and stationary contact for each

cusp; and we see that there is no difficulty in completing the investigation—that is, being given the degree, the finite double points or cusps, and the double points or cusps at the circular points at infinity of a plane curve, to find the characteristics of the evolute.

CHAPTER XII.

Sphero-Cartesians.

235. If a Cartesian cyclide be intersected by any sphere, I shall call the curve of intersection a *sphero-Cartesian.* It is evident, if the intersecting sphere become a plane, that the sphero-Cartesian will become a Cartesian oval. We have seen that, being given a sphere U and a quadric F, the cyclide which has U for a sphere of inversion and F for a focal quadric will intersect U in the same sphero-quartic as the reciprocal of F with respect to U intersects U. Now, when F is a sphere its reciprocal with respect to U is a quadric of revolution. Hence we have the following fundamental theorem:—

A sphero-Cartesian is the curve of intersection of a sphere and a quadric of revolution.

236. *The focal sphero-conics of a sphero-Cartesian are circles.*

Demonstration. Let the sphero-Cartesian be the intersection of the sphere U and quadric V. Then, since V is a quadric of revolution, the cones which can be described through (UV) have but one system of circular sections, and therefore the cones reciprocal to them have but one system of focal lines; but the reciprocal cones with respect to U intersect U in the focal sphero-conics of UV; therefore the focal sphero-conics of UV are circles.

237. *One of the four cones through* (UV) *is a right cylinder on a parabolic base, the plane of the base being perpendicular to the planes of circular sections of* V.

Demonstration. Let

$$U\equiv(x+\alpha)^2+(y+\beta)^2+(z+\gamma)^2-r^2=0,$$

$$V\equiv\left(\frac{x-\alpha}{a}\right)^2+\left(\frac{y-\beta}{a}\right)^2+\left(\frac{z-\gamma}{c}\right)^2-1=0;$$

then

$$U-a^2V\equiv 4\alpha x+4\beta y+(z+\gamma)^2-\frac{a^2}{c^2}(z-\gamma)^2+a^2-r^2=0,$$

and this will be of the form

$$a'z^2+f'y+d=0$$

by a change of axes. Hence the proposition is proved; or we may show it thus: the biquadratic

$$\frac{\alpha^2}{a^2+\lambda}+\frac{\beta^2}{b^2+\lambda}+\frac{\gamma^2}{c^2+\lambda}=1+\frac{\varrho^2}{\lambda}$$

(see SALMON'S 'Geometry of Three Dimensions,' page 146), whose roots are the values of λ, for which

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2-\varrho^2+\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)=0$$

represents a cone, reduces to a cubic when $a=b$, showing that in this case there are only three cones.

Again, the equation

$$\frac{\lambda a\alpha^2}{1+a\lambda}+\frac{\lambda b\beta^2}{1+b\lambda}+2\lambda r\gamma=\lambda^2r^2+\varrho^2,$$

which is the biquadratic for the paraboloid $ax^2+by^2+2rz=0$ and the sphere

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2-\varrho^2,$$

becomes a cubic when $a=b$.

238. *If a sphero-Cartesian be projected on the plane of circular section of* V *by lines parallel to the axis of revolution, the projection will be a Cartesian oval.*

Demonstration. Let $\mathrm{U}\equiv x^2+y^2+z^2-r^2=0$,

$$\mathrm{V}\equiv\frac{(x-\alpha)^2}{a^2}+\frac{(y-\beta)^2}{a^2}+\frac{(z-\gamma)^2}{c^2}-1=0.$$

Now, putting $x^2+y^2-r^2\equiv-\mathrm{S}$ and $c^2\left(1-\frac{(x-\alpha)^2+(y-\beta)^2}{a^2}\right)\equiv\mathrm{S}'$, these equations are equivalent to $z-\mathrm{S}^{\frac{1}{2}}=0$, $z-\gamma-\mathrm{S}'^{\frac{1}{2}}=0$. Hence, eliminating z, we get the equation of a Cartesian oval.

239. *If a plane parallel to the planes of circular section of* V *intersect* U *and* V *in two circles* u *and* v, *the locus of the radical axis of* u *and* v *will be the cylinder on the parabolic base.*

Demonstration. Put $z=k$ in the equations of U and V, and we have their sections by the plane $z=k$; thus

$$u\equiv x^2+y^2+k^2-r^2=0,\quad v\equiv\frac{(x-\alpha)^2}{a^2}+\frac{(y-\beta)^2}{a^2}+\frac{(k-\gamma)^2}{c^2}-1=0.$$

Hence the radical axis of u and v is $u-a^2v=0$; therefore the same value of λ for which $\mathrm{U}+\lambda\mathrm{V}$ becomes a parabolic cylinder reduces $u+\lambda v$ to the radical axis of u and v, and the proposition is proved.

Cor. 1. The curve of intersection of a sphere with a cylinder on a parabolic base is a sphero-Cartesian.

Cor. 2. From recent articles we infer the following method of generating sphero-Cartesians.

Let J be a circle and F a parabola in the same plane (say, in the plane of the paper); then from any point P in F erect two perpendiculars in opposite directions to the plane of the paper and equal respectively to $\pm\mathrm{T}\sqrt{-1}$, where T is the length of the tangent drawn from P to J; then the locus of the extremities of the perpendicular will be a sphero-Cartesian.

240. Since one of the four cones passing through the sphero-Cartesian (UV) is a parabolic cylinder, it follows that one of the nodal conics of the developable Σ formed by tangent planes to U along (UV) will pass through the centre of U. Hence we have the following theorem:—

The binodal cyclide $(a, b, c, f, g, h)(\alpha, \beta, \gamma)^2=0$ *will be intersected by the sphere* U *orthogonal to* α, β, γ, *and whose centre is complanar with their centres in a sphero-Cartesian if the conic* $(a, b, c, f, g, h)(\lambda, \mu, \nu)^2$ *pass through the centre of* U.

241. From the method of generating sphero-Cartesians given in art. 239, *Cor.* 2, we can get one form of its equation considered as a curve described on a sphere.

Thus, let the equation of the sphere of which the circle J is a great circle be $x^2+y^2+z^2=1$, and the equation of F be $(y+k)^2=4a(h+x)$, or, in polar coordinates,

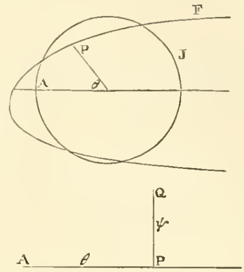
$$(\rho \sin\theta+k)^2=4a(h+\rho\cos\theta),$$

and it is clear that the perpendicular to the plane of the paper at P will cut the sphere in a point Q whose spherical coordinates are thus determined.

Taking the great circle J as a circle of reference, making $\text{AP}=\theta$, PQ perpendicular to it $=\psi$, then we have $\cos\psi=\rho$, and the equation required is

$$(\sin\theta\cos\psi+k)^2=4a(h+\cos\theta\cos\psi). \quad . \quad . \quad . \quad (149)$$

Fig. 6.



242. Let the great circle J of the sphere intersect the parabolic base of the cylinder in four points, and let K, K′, K″ be the points of intersection of the three pairs of lines through these four points, the sides of the triangle K K′ K″ will cut off from the sphere three arcs, and the three small circles which have these three arcs as spherical diameters will be the three circles of inversion of the sphero-Cartesian. Again, the three pairs of perpendiculars from the centre of the sphere on the three pairs of opposite connectors will cut the sphere in three pairs of points which will be the extremities of the diameters of the three focal circles of the sphero-Cartesian. Hence, being given on the surface of a sphere U a focal circle F and a circle of inversion J of a sphero-Cartesian, we infer the following construction for the two remaining focal circles and circles of inversion:—

Let H *and* O *be the centres of* J *and* F, *and let* HO *intersect the circles* J *and* F *in the points* A, B, C, D; *if the points* P *and* Q *be taken so as to be common points of harmonic section of* AB *and* CD, *then* P *and* Q *are plainly the points in which radii from the centre of* U *to the points* K′, K″ *pierce* U; *they are therefore the centres of inversion of the sphero-Cartesian.*

243. Again (see fig. 5, art. 198), if the circles F′, F″ be described on the sphere as they are in the diagram referred to on the plane, we shall have the three focal circles and their radii given by the following equations:—

$$\left.\begin{aligned}\tan^2 r &= \tan \text{PH}\,.\,\tan \text{QH},\\ \tan^2 r' &= \tan \text{OH}\,.\,\tan \text{QH},\\ \tan^2 r'' &= \tan \text{OH}\,.\,\tan \text{PH}.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (150)$$

The first equation is evident, since P, Q are conjugate points with respect to J; and

the second follows from the fact that the bisectors of the angles ABD and DBN pass respectively through O and H.

244. The points O, P, Q are the centres of J, J′, J″. Let their distances from H be denoted by δ, δ', δ'' respectively, and the preceding equations may be written

$$\left.\begin{aligned}\tan^2 r &= \tan\delta' \tan\delta'',\\ \tan^2 r' &= \tan\delta'' \tan\delta,\\ \tan^2 r'' &= \tan\delta \tan\delta'.\end{aligned}\right\} \quad (151)$$

Hence we get the three following equations:

$$\left.\begin{aligned}\tan\delta &= \tan r' \tan r'' : \tan r,\\ \tan\delta' &= \tan r'' \tan r : \tan r',\\ \tan\delta'' &= \tan r \tan r' : \tan r''.\end{aligned}\right\} \quad (152)$$

Hence also we get

$$\tan\delta \tan\delta' \tan\delta'' = \tan r \tan r' \tan r''. \quad (153)$$

245. If we denote the radii of the circles of inversion J, J′, J″ by ϱ, ϱ', ϱ'', we easily get the system of three equations,

$$\left.\begin{aligned}\tan^2\varrho &= \tan(\delta-\delta')\tan(\delta-\delta''),\\ \tan^2\varrho' &= \tan(\delta'-\delta'')\tan(\delta'-\delta),\\ \tan^2\varrho'' &= \tan(\delta''-\delta)\tan(\delta''-\delta'),\end{aligned}\right\} \quad (154)$$

with this other system of equations,

$$\left.\begin{aligned}\tan(\delta-\delta') &= \tan\varrho \tan\varrho' : \tan\varrho''\sqrt{-1},\\ \tan(\delta'-\delta'') &= \tan\varrho' \tan\varrho'' : \tan\varrho'\sqrt{-1},\\ \tan(\delta''-\delta) &= \tan\varrho'' \tan\varrho : \tan\varrho''\sqrt{-1}.\end{aligned}\right\} \quad (155)$$

Hence also

$$\tan(\delta-\delta')\tan(\delta'-\delta'')\tan(\delta''-\delta') = \tan\varrho \tan\varrho' \tan\varrho''\sqrt{-1}. \quad (156)$$

From either system it follows, as we know otherwise, that one of the circles of inversion is imaginary.

246. From combining the equations of the two preceding articles, we easily get the relations between the radii of the F's and the J's; thus

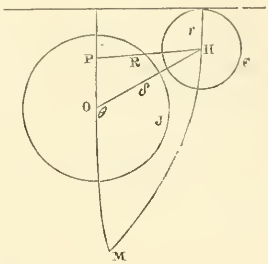
$$\begin{aligned}\tan^2\varrho &= \tan(\delta-\delta')\tan(\delta-\delta'')\\ &= \frac{\tan^2\delta - \tan\delta(\tan\delta' + \tan\delta'') + \tan\delta' \tan\delta''}{(1+\tan\delta \tan\delta')(1+\tan\delta \tan\delta'')}\\ &= \frac{(\tan^2 r - \tan^2 r')(\tan^2 r - \tan^2 r'')}{\tan^2 r \sec^2 r' \sec^2 r''}\\ &= \frac{4(\sin^2 r - \sin^2 r')(\sin^2 r - \sin^2 r'')}{\sin^2 2r}.\end{aligned}$$

Hence we have the system of equations:

$$\left.\begin{aligned}\tan^2\varrho\ &=4(\sin^2 r\ -\sin^2 r'\ )(\sin^2 r\ -\sin^2 r'') : \sin^2 2r\ ,\\ \tan^2\varrho' &=4(\sin^2 r' -\sin^2 r'')(\sin^2 r' -\sin^2 r\ ) : \sin^2 2r',\\ \tan^2\varrho'' &=4(\sin^2 r''-\sin^2 r\ )(\sin^2 r''-\sin^2 r'\ ) : \sin^2 2r''.\end{aligned}\right\} \quad . \quad . \quad . \quad (157)$$

247. Let us denote the radii of the three circles of inversion by J, J′, J″, the radii of the focal circles by r, r', r'', and the distances by δ, δ', δ'', as in recent articles. Now denoting the perpendicular from the centre of J to any tangent to F by p, then taking $\mathrm{OP}=\varrho$ such that

Fig. 7.



$$\cos p : \cos(p-\varrho)=\cos \mathrm{J}=\frac{1}{k} \text{ suppose ;}$$

$$\therefore\ (\cos\varrho-k)\cos p+\sin\varrho\sin p=0\ ;$$

but from the spherical triangles O H M and O P H we get

$$\cos p\sin\delta\cos\theta+\cos\delta\sin p=\sin r,$$

$$\cos\varrho\cos\delta+\sin\varrho\sin\delta\cos\theta=\cos \mathrm{R}.$$

Hence, eliminating p and θ, we get

$$(1+k^2-2k\cos\varrho)^{\frac{1}{2}}\sin r=k\cos\delta-\cos \mathrm{R},$$

with two similar expressions involving k', k''; δ', δ''. Hence we have the determinant

$$\begin{vmatrix} (1+k^2\ -2k\ \cos\varrho\ )^{\frac{1}{2}}\sin r\ , & k\ \cos\delta\ , & 1,\\ (1+k'^2\ -2k'\ \cos\varrho'\ )^{\frac{1}{2}}\sin r', & k'\cos\delta', & 1,\\ (1+k''^2-2k''\cos\varrho'')^{\frac{1}{2}}\sin r'', & k''\cos\delta'', & 1, \end{vmatrix}=0\ ;$$

and restoring the value of k, k', k'', we get

$$\begin{vmatrix} (1+\cos^2 \mathrm{J}\ -2\cos\varrho\cos \mathrm{J}\ )^{\frac{1}{2}}\sin r\ , & \cos\delta\ , & \cos \mathrm{J}\ ,\\ (1+\cos^2 \mathrm{J}'\ -2\cos\varrho'\cos \mathrm{J}'\ )^{\frac{1}{2}}\sin r', & \cos\delta', & \cos \mathrm{J}',\\ (1+\cos^2 \mathrm{J}''-2\cos\varrho''\cos \mathrm{J}'')^{\frac{1}{2}}\sin r'', & \cos\delta'', & \cos \mathrm{J}'', \end{vmatrix}=0. \quad . \quad . \quad (158)$$

248. It is evident that $1+k^2-2k\cos\varrho$ is equal to the square of the distance of the point P of the sphero-Cartesian from the pole of the plane of J with respect to the sphere U. Hence, if this distance be denoted by D, the first determinant of the last article may be written

$$\begin{vmatrix} \mathrm{D}\sin r\ , & \cos\delta\sec \mathrm{J}\ , & 1,\\ \mathrm{D}'\sin r', & \cos\delta'\sec \mathrm{J}', & 1,\\ \mathrm{D}''\sin r'', & \cos\delta''\sec \mathrm{J}'', & 1, \end{vmatrix}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (159)$$

249. From the equation, art. 247,

$$(1+k^2-2k\cos\varrho)^{\frac{1}{2}}\sin r=k\cos\delta-\cos R,$$

which we may put in the form

$$(1+k^2-2k\cos\varrho)^{\frac{1}{2}}\sin r=C,$$

where C is the small circle cutting J orthogonally and concentric with F, we have this other equation,

$$(k-e^{\varrho\sqrt{-1}})(k-e^{-\varrho\sqrt{-1}})\sin^2 r=C^2. \quad \text{(160)}$$

Hence the imaginary lines $k-e^{\varrho\sqrt{-1}}$, $k-e^{-\varrho\sqrt{-1}}$ are tangents to the sphero-Cartesian. Hence the centre of J is a focus; and similarly the centres of J', J'' are foci.

250. The equation

$$(1+k^2-2k\cos\varrho)\sin^2 r=C^2$$

is the envelope of the circle

$$1+k^2-2k\cos\varrho+\mu C+\mu^2\sin^2 r=0\,;$$

but

$$C=k\cos\delta-(\cos\varrho\cos\delta-\sin\varrho\sin\delta\cos\theta)=0,$$

$$\therefore 1+k^2+\mu k\cos\delta+\mu^2\sin^2 r=(2k+\mu\cos\delta)\cos\varrho-\mu\sin\delta\sin\varrho\cos\theta.$$

Now the equation of a circle (see art. 36) is

$$\cos R=\cos n\cos\varrho+\sin n\sin\varrho\cos\theta.$$

Multiplying by an indeterminate constant, we get

$$\lambda\cos R=1+k^2+\mu k\cos\delta+\mu^2\sin^2 r,$$

$$\lambda\cos n=2k+\mu\cos\delta,$$

$$\lambda\sin n=-\mu\sin\delta.$$

Hence

$$\cos R=\frac{1+k^2+\mu k\cos\delta+\mu^2\sin^2 r}{\sqrt{\mu^2+4\mu k\cos\delta+4k^2}}. \quad \text{(161)}$$

Now if R be equal to zero, the circle whose radii is R will be a focus, since it will have imaginary double contact with the sphero-Cartesian; but $R=0$ gives the biquadratic in μ,

$$(1+k^2+\mu k\cos\delta+\mu^2\sin^2 r)^2=(\mu^2+4\mu k\cos\delta+4k^2), \quad \text{(162)}$$

showing that there are four foci, as we know otherwise, since there are three single foci and one triple focus.

251. If $R=\frac{\pi}{2}$, we have the equation

$$1+k^2+\mu k\cos\delta+\mu^2\sin^2 r=0, \quad \text{(163)}$$

a quadratic showing that a sphero-Cartesian has two great circles which have double contact with it. These are not, however, the only great circles which have double contact with the sphero-Cartesian. These correspond to the great circle passing

through the three single foci of the curve; and we will now show that there are two great circles of double contact corresponding to each of the three small circles of inversion, J, J′, J″. For let F be the focal circle corresponding to J, then the great circle whose pole is the centre of J will intersect F in two points; these will be the poles of two great circles, each having double contact with the sphero-Cartesian. Hence a *sphero-Cartesian has eight great circles of double contact.*

Cor. *It is evident that a similar property holds for a sphero-quartic.*

252. If in the equation of art. 250

$$1+k^2+\mu k\cos\delta+\mu^2\sin^2 r=(2k+\mu\cos\delta)\cos\varrho-\mu\sin\delta\sin\varrho\cos\theta$$

we substitute the spherical coordinates ϱ', θ' of any point (see art. 36), we get a quadratic for μ, showing that through any point ϱ', θ' two generating circles pass. Hence, reasoning as in the last article, *through any point may be described eight circles each having double contact with the sphero-Cartesian.* Hence, if we invert the sphere into a plane, *the inverse of the sphero-Cartesian will not be a Cartesian oval but a bicircular quartic.*

253. The following properties of sphero-Cartesians are the analogues of properties of plane Cartesians which have appeared in the 'Educational Times':—

1°. *Being given two small circles such that a spherical triangle can be inscribed in one and circumscribed to the other, the envelope of the small circle which has the spherical triangle as a self-conjugate, or, as it may more appropriately be called, an harmonic triangle, is a sphero-Cartesian.*

2°. *Through any point on a sphero-conic can be described three circles which osculate the sphero-conic; the envelope of the circle through the three points of osculation is a sphero-quartic.*

3°. *If a sphero-quartic with a double point* O *be cut by a circle in four points* A, B, C, D, *and if* OA, OB, OC, OD *cut the circle again in* E, F, G, H, *any pair of great circles through these points will be equally inclined to the bisectors of the angles between the tangents at* O.

4°. *If a sphero-conic be turned through* 90° *round the principal axis of the cone which cuts the sphere in the sphero-conic, the locus of the intersection of any tangent with the same tangent in its new position is a sphero-quartic.*

5°. The locus of one set of foci of all the conics which have double contact with a given circle at given points is another circle passing through those points and through the centre of the given circle. Hence, by inversion, *the locus of one set of foci of sphero-quartics with a double point which have a given generating circle, and which have given points of contact with it, is a circle through the points of contact.*

6°. The three points in which a circular cubic is cut by any transversal are the foci of a Cartesian oval passing through four concyclic foci of the cubic. Hence, by inversion, *four concyclic points on a sphero-quartic* A *are the foci of another sphero-quartic* B *passing through four concyclic foci of* A. It is evident that this property is analogous to that of pole and polar, and that a similar use may be made of it.

254. The following properties are the inverses of properties of conics &c. :—

1°. A circular cubic is the locus of one set of foci of all the conics that can be drawn through four concyclic points. Hence, by inversion, a *sphero-quartic is the locus of the locus of one set of foci of all the sphero-quartics with a double point which can be drawn through four concyclic points.*

A more general proposition than this can be easily inferred from art. 253, 6°.

2°. If two tangents to a conic intersect at a given angle, the locus of their intersection is a bicircular quartic. Hence, by inversion,

If two generating circles of a sphero-quartic with a double point (including cusps and conjugate points) intersect at a given angle, the locus of their intersection, if they belong to the system of generating circles which passes through the double point, is a sphero-quartic.

Cor. If the angle of intersection be a right angle the locus will be a circle.

3°. A cardioide can be inverted into a cissoid. *Hence a cusped sphero-quartic will be got by inverting a cusped sphero-Cartesian.*

255. Particular spherical sections of a general cyclide will be sphero-Cartesians.

The following is an example:—*Let* W *be a cyclide,* U *and* F *a sphere of inversion and corresponding focal quadric; then if any sphere has its centre on the focal hyperbola of* F *and cuts* U *orthogonally, it will intersect* W *in a sphero-Cartesian.*

CHAPTER XIII.

SECTION I.—*Substitutions.*

256. If $W=(a, b, c, d, l, m, n, p, q, r \between \alpha, \beta, \gamma, \delta)^2=0$ be the equation of a cyclide, and if the equation W be satisfied by the values x', y', z', w' of $\alpha, \beta, \gamma, \delta$, we can state it thus: the system of six spheres denoted by the matrix

$$\left\| \begin{matrix} \alpha, & \beta, & \gamma, & \delta, \\ x', & y', & z', & w', \end{matrix} \right\| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (164)$$

have the two points which are common to them on W; and if the ratios $x' : y' : z' : w'$ be supposed to vary, but subject to the condition

$$(a, b, c, d, l, m, n, p, q, r \between x', y', z', w')^2=0,$$

then the pair of points denoted by the matrix (164) will vary, and the locus will be the cyclide W. Hence we may call $(a, b, c, \ldots . r \between \alpha, \beta, \gamma, \delta)^2=0$ the local equation of a cyclide.

I remark that whenever I shall speak of a pair of inverse points on a cyclide it will be a pair determined by a matrix such as (164).

257. We have seen that the tangential equation of the focal quadric of a cyclide is the same in form as the local equation (see last article) of the cyclide, and that to a tangent plane of the quadric will correspond a pair of inverse points of the cyclide, and generally

to any plane L related to the quadric will correspond a pair of inverse points having a correlative reference to the cyclide, and these inverse points will be the limiting points of the sphere U (the Jacobian of α, β, γ, δ) and the plane L.

258. We have determined in art. 5 the condition that the sphere $x\alpha+y\beta+z\gamma+w\delta$ should be a generating sphere of W to be given by the determinant (7), and that this determinant in tetrahedral coordinates is the equation of the focal quadric F of W. Now since for any system of values of x, y, z, w which satisfies the determinant (7) we get a point on F, we see that to any point on F will correspond a generating sphere of W, and generally to any point P having any special relation to F will correspond a sphere Q having a similar relation to W; in fact the sphere Q will have the point P for centre, and will be orthogonal to U.

259. Since the tetrahedral coordinates of the centre of $x_1\alpha+y_1\beta+z_1\gamma+w_1\delta$ are x_1, y_1, z_1, w_1, and if four spheres orthogonal to U pass through the same pair of inverse points, with respect to U we know that their centres are complanar. Hence we have the following theorem:—

The condition that the four spheres

$$x_1\alpha+y_1\beta+z_1\gamma+w_1\delta,\ x_2\alpha+y_2\beta+z_2\gamma+w_2\delta,\ \&c.$$

should pass through the same pair of inverse points is the vanishing of the value of the determinant

$$\begin{vmatrix} x_1, & y_1, & z_1, & w_1, \\ x_2, & y_2, & z_2, & w_2, \\ x_3, & y_3, & z_3, & w_3, \\ x_4, & y_4, & z_4, & w_4. \end{vmatrix} \quad \dots\dots\dots\dots \quad (165)$$

260. In art. 257 it is proved that the pair of inverse points given by the matrix (164) correspond to a plane, and in art. 27 it is shown that the perpendiculars from the centres of $\alpha, \beta, \gamma, \delta$ on the plane are proportional to x_1, y_1, z_1, w_1 of the matrix, that is, in other words, the coordinates of the plane are x_1, y_1, z_1, w_1. Hence we infer the following theorem:—

The four pairs of inverse points given by the matrices

$$\left\|\begin{matrix} \alpha, & \beta, & \gamma, & \delta, \\ x_1, & y_1, & z_1, & w_1, \end{matrix}\right\|; \quad \left\|\begin{matrix} \alpha, & \beta, & \gamma, & \delta, \\ x_2, & y_2, & z_2, & w_2, \end{matrix}\right\|; \quad \left\|\begin{matrix} \alpha, & \beta, & \gamma, & \delta, \\ x_3, & y_3, & z_3, & w_3, \end{matrix}\right\|; \quad \left\|\begin{matrix} \alpha, & \beta, & \gamma, & \delta, \\ x_4, & y_4, & z_4, & w_4, \end{matrix}\right\|$$

are homospheric, if vanishes the determinant

$$\begin{vmatrix} x_1, & y_1, & z_1, & w_1, \\ x_2, & y_2, & z_2, & w_2, \\ x_3, & y_3, & z_3, & w_3, \\ x_4, & y_4, & z_4, & w_4. \end{vmatrix} \quad \dots\dots\dots\dots \quad (166)$$

261. Since to any system of complanar points corresponds a system of spheres passing through a pair of inverse points, *to a plane conic on the focal quadric of a cyclide* W *will*

correspond a binodal cyclide circumscribed about W; *the nodes of the binodal cyclide will correspond to the plane of the conic.*

262. Since to any system of planes passing through a point corresponds a system of homospheric pairs of inverse points, *to a cone circumscribed about the focal quadric of a cyclide* W *will correspond a sphero-quartic on* W; *and if the cone be one of revolution, the sphero-quartic will become a sphero-Cartesian.*

Cor. 1. *If the vertex of the cone be at infinity, that is, if the cone become a cylinder, to it will correspond a bicircular quartic; and if the cylinder be one of revolution, to it will correspond a section of the cyclide, which will be a Cartesian oval.*

Cor. 2. *Since two cylinders of revolution can be described about a quadric, through each centre of inversion of a cyclide can be drawn two planes which will intersect it in Cartesian ovals.*

263. Since to a point on F corresponds a generating sphere of W, to the line joining two points on F will correspond the circle of intersection of two generating spheres; and if every point of the line be on F, every point of the circle will be on W. Hence to a rectilinear generator of F will correspond a circular generator of W; and since through any point on F can be drawn two rectilinear generators, *hence in general can be drawn two circular generators corresponding to each focal quadric of* W *through any point of* W.

264. The last article may be established differently as follows. Thus if perpendiculars from the centres of the spheres of reference α, β, γ, δ of the cyclide $W = a\alpha^2 + b\beta^2 + c\gamma^2 + d\delta^2$ on any plane be denoted by λ, μ, ν, ϱ, then the points whose equations are

$$\begin{vmatrix} A, & B, & C, & D, \\ A', & B', & C', & D', \end{vmatrix} (\lambda, \mu, \nu, \varrho)$$

correspond to the spheres whose equations are

$$\begin{vmatrix} A, & B, & C, & D, \\ A', & B', & C', & D', \end{vmatrix} (\alpha, \beta, \gamma, \delta),$$

and consequently the line joining the points corresponds to the circle of intersection of the spheres. Now the six determinants of the matrix

$$\begin{Vmatrix} A, & B, & C, & D, \\ A', & B', & C', & D', \end{Vmatrix}$$

or their mutual ratios, are called by Professor CAYLEY the six coordinates of a line in space, and are denoted by the notation (a, b, c, f, g, h). Hence we see that we can in our extension call the same ratios the six coordinates of a circle in space (see CAYLEY "On the Six Coordinates of a Line," Cambridge Philosophical Transactions, vol. xi. pt. 2). Hence the same investigation which in Professor CAYLEY's system proves any property of a system of lines in space, will, with our interpretation, give a corresponding property of a system of circles in space. It is to be remembered, however, that all our circles are cut orthogonally by the same sphere, namely the sphere U, the Jacobian of α, β, γ, δ

(compare art. 19). As an example, three lines in space determine a ruled quadric; to which we have the corresponding theorem:—Three circles in space orthogonal to the same sphere determine a cyclide. Again, every ruled quadric has two systems of rectilinear generators; to this corresponds the theorem:—Every cyclide has two systems of circular generators corresponding to each sphere of inversion; or any four rectilinear generators of one system on a ruled quadric are cut equianharmonically by all the rectilinear generators of the opposite. Hence any four circular generators of one system belonging to a cyclide are cut equianharmonically by all the circular generators of the opposite system.

265. From recent articles we see that, being given any graphic property of the focal quadric F of a cyclide W, we can get a corresponding property of W by the following substitutions:—

	F	W
I.	*a.* For a point on F, ...	*a.* A generating sphere of W.
	b. A point having any special relation to F, ...	*b.* A sphere having a corresponding relation to W.
	c. A system of complanar points,	*c.* A system of spheres through the same two points.
	d. A system of collinear points, .	*d.* A system of spheres through the same circle.
II.	*a.* A tangent line to	*a.* A circle having double contact with
	b. A line having any special relation to	*b.* A circle having a corresponding relation to
	c. A system of concurrent lines,	*c.* A system of homospheric circles.
	d. A system of complanar lines,	*d.* A system of circles through the same two points.
III.	*a.* A tangent plane to	*a.* A pair of inverse points on
	b. A plane having any special relation to	*b.* A pair of inverse points having a corresponding relation to
	c. A system of planes through the same point,	*c.* A system of homospheric pairs of inverse points.
	d. A system of planes through the same line,	*d.* A system of inverse pairs of points on the same circle.

266. In order to give illustrations of this system of substitutions, I give the theorems derived by them from a splendid paper of Mr. Townsend's in the 'Quarterly Journal,' vol. viii. p. 10. For this purpose the following proposition is necessary:—If three tangent planes to a quadric be mutually perpendicular the locus of their point of intersection is a sphere, called the director sphere of the quadric. Hence, by substitution, we get the following theorem:—*If three lines mutually perpendicular be drawn through a centre of inversion of a cyclide, and if* P, P′, Q, Q′, R, R *be three pairs of inverse points*

in which these lines intersect the cyclide, the envelope of the sphere through P, P′, Q, Q′, R, R′ *is a Cartesian cyclide, which by analogy I shall call the director cyclide of the given cyclide.*

1°. "If a system of quadrics touch a common system of eight planes, their director spheres have a common radical plane." Hence, *if a system of cyclides pass through a common system of eight inverse pairs of points, their director Cartesian cyclides are inscribed in a common binodal Cartesian cyclide.*

2°. "If a system of quadrics touch a common system of seven planes, their director spheres have a common radical axis." Hence, *if a system of cyclides pass through a common system of seven inverse pairs of points, their director Cartesians have two generating spheres common to all.*

3°. "If a system of quadrics touch a common system of six planes, their director spheres have a common radical centre." Hence, *if a system of cyclides pass through a common system of six inverse pairs of points, their director Cartesian cyclides are such that any of them can be expressed as a linear function of four others*, because their director spheres in the property of the quadrics having a common radical centre are coorthogonal, and any of them can be expressed as a linear function of four others.

Cor. The property in 1° may be expressed thus:—any director Cartesian can be expressed linearly in terms of two others; and the property in 2°, any director Cartesian can be expressed linearly in terms of three others.

267. When a quadric becomes a paraboloid, the director sphere becomes a director plane. Hence if three lines mutually perpendicular be drawn through a centre of inversion of a cubic cyclide intersecting it in three pairs of inverse points P, P′, Q, Q′, R, R′, the sphere determined by these three pairs of inverse points passes through a fixed pair of points. *I shall call these points the director points of the cubic cyclide.*

1°. "If a system of quadrics touch the same eight planes, the common radical plane of their director spheres is the director plane of the paraboloid which touches the planes." Hence, *if a system of cyclides pass through a common system of eight inverse pairs of points, the nodes of the binodal Cartesian cyclide which is circumscribed to their director Cartesian cyclides are the director points of the cubic cyclide which passes through the system of eight pairs of inverse points.*

2°. "If a system of paraboloids touch the same seven planes, their director planes have a common line of intersection." Hence, *if a system of cubic cyclides pass through a common system of seven pairs of inverse points, their director points are concyclic.*

3°. "If a system of quadrics having a common rectilinear generator touch five planes, their director spheres have a common radical plane." Hence, *if a system of cyclides having a common circular generator pass through five inverse pairs of points, their director cyclides are inscribed in the same binodal Cartesian cyclide.*

4°. "If a system of ruled quadrics have two common rectilinear generators and touch two common planes, their director spheres have a common circle of intersection with that of the ruled quadric passing through the two lines and through the intersection of

the two planes." Hence, *if a system of cyclides having two common circular generators pass through two inverse pairs of points, their director cyclides are inscribed in a binodal Cartesian cyclide, in which is also inscribed the cyclide determined by the two circular generators and the circle through the two inverse pairs of points.*

5°. "The director sphere of every ruled quadric passing through the four sides L, M, N, P of any skew quadrilateral passes through the circle of intersection of the two spheres of which the two diagonals are diameters." Hence, *if* L, M, N, P *be four circles in space, such that each of the four pairs* (LM), (MN), (NP), (PL) *is homospheric, and if* L, M, N, P *be circles on a cyclide* W, *the director cyclide of* W *can be expressed as a linear function of two Cartesian cyclides, viz. the cyclide which has the spheres* (LM) and (NP) *as generating spheres, and the middle point between their centres as a triple focus, and the cyclide similarly determined by the spheres* (MN) *and* (PL).

6°. "When eight planes pass in pairs through any generators of the same ruled quadric, the director spheres of all quadrics touching them all have a common circle of intersection with that of the original quadric." Hence, *when eight pairs of inverse points lie two by two on four circular generators of a given cyclide, the director cyclides of all cyclides passing through the eight pairs are such that the director cyclide of the given cyclide can be expressed as a linear function of any two of them; in other words, the director cyclides of the variable cyclides and the director cyclides of the given cyclides are all inscribed in the same binodal Cartesian cyclide.*

7°. *If a system of paraboloids touch the same six planes, their director planes have a common point of intersection.* Hence, *if a system of cubic cyclides pass through the same six pairs of inverse points, all their director pairs of points are homospheric.*

8°. "If a system of ruled quadrics passing through a common line touch four common planes, their director spheres have a common radical axis." Hence, *if a system of cyclides having a common circular generator pass through four inverse pairs of common points, their director cyclides have two common generating spheres.*

268. I shall for the next illustration take the properties of quadrics given in a paper of Dr. Salmon's in the same volume of the Quarterly Journal, "On the Number of Surfaces of the Second Degree which satisfy nine conditions."

1°. Dr. Salmon proves "that two quadrics can be described through eight given points to touch a given line." Hence *two cyclides having a given sphere of inversion can be described having eight given generating spheres to have double contact with a given circle.*

2°. "Three quadrics can be described through eight given points to touch a given plane." Hence *three cyclides can be described having eight given generating spheres to pass through a given pair of inverse points.*

3°. "Twenty-one quadrics can be described through five given points to touch four planes." Hence *twenty-one cyclides can be described having five given generating spheres to pass through four given inverse pairs of points.*

4°. *In general if* N(r, s, t) *denote the number of quadrics which can be described to pass*

through r *points, to touch* s *lines, and to touch* t *planes, where* r+s+t=9, *then precisely the same number* N *of cyclides can be described, being given* r *generating spheres to have double contact with* s *circles and to pass through* t *inverse pairs of points.*

269. If V be the reciprocal of the focal quadric F with respect to U, or, in other words, if V be a quadric of the system passing through the sphero-quartic WU, then the planes, lines, and points of V will correspond to the points, lines, and planes of F; and hence by substitutions reciprocal to those of art. 265, being given any graphic property of V, we can get a corresponding graphic property of W.

Cor. Hence, being given any graphic of a quadric, we can get two correlative graphic properties of a cyclide.

SECTION II.—*Substitutions. Sphero-quartics.*

270. We have seen, if in the equation of a cyclide $W=\phi(\alpha, \beta, \gamma, \delta)$, where ϕ represents a homogeneous function of the second degree, we regard $\alpha, \beta, \gamma, \delta$ as the small circles in which the spheres $\alpha, \beta, \gamma, \delta$ intersect U, that we get the equation of the sphero-quartic (WU); also that the sphero-quartic is generated as the envelope of a variable circle, whose centre moves along a sphero-conic, and which cuts a given circle orthogonally; and we might investigate, as in the last section, a system of substitutions by which, from known properties of sphero-conics, we could infer properties of sphero-quartics; but there is a simpler system of substitutions by which we may arrive at the latter, namely, by means of substitutions from known properties of plane conics. This method I shall explain briefly in the following articles.

271. Let $W=a\alpha^2+b\beta^2+c\gamma^2+d\delta^2=0$ be a cyclide, then the Jacobian of $\alpha, \beta, \gamma, \delta$ is given by the equation

$$U^2=\alpha^2+\beta^2+\gamma^2+\delta^2=0.$$

Hence the sphero-quartic (WU) will be the curve of intersection of U, and either of the binodal cyclides

$$W-aU^2, \quad W-bU^2, \quad W-cU^2, \quad W-dU^2.$$

Now let us consider $W-aU^2$, or $(b-a)\beta^2+(c-a)\gamma^2+(d-a)\delta^2$; this cyclide has four focal quadrics, of which one reduces to a plane conic, and this conic is a focal conic of each of three remaining focal quadrics (see art. 113). The conic is one of the nodal lines of the developable Σ (see Chapter VIII.), and is the reciprocal of one of the four cones through (WU). Now any tangent plane to the cone will intersect U in a circle, which will be a generating circle of WU, and this tangent plane will intersect the plane of the nodal conic of Σ, that is, the plane of the conic whose equation in tangential coordinates is

$$(b-a)\mu^2+(c-a)^2\nu+(d-a)\varrho^2,$$

in a line, which will be a tangent line to the reciprocal conic, that is, to the conic whose trilinear equation is $(b-a)y^2+(c-a)z^2+(d-a)w^2=0$. Hence a tangent line to this conic corresponds to a generating circle of WU. Again, any edge of the cone intersects the conic $(b-a)y^2+(c-a)z^2+(d-a)w^2$ in a point, and passes through a pair of points

of WU; this pair of points will be inverse to each other with respect to the vertex of the cone. Hence a point on the conic corresponds to a pair of inverse points on the sphero-quartic. Again, a point and its polar with respect to the conic correspond to a pair of inverse points and a circle, which are related to each other with respect to the sphero-quartic, as poles and polars are in ordinary geometry. For example, the point and polar with respect to the conic are such that any line through the point meets the conic in two points such that tangents to the conic drawn through them meet on the polar; and the pair of inverse points and their polar circle are such that any circle through the inverse points meets the sphero-quartic in two pairs of points, such that the generating circles which touch the sphero-quartic at these points intersect on the polar circle of the pair of inverse points.

If we have any system of collinear points on the plane of $(b-a)y^2+(c-a)z^2+(d-a)w^2$, it is evident we shall have to correspond with them a system of inverse pairs of points which are concyclic. Lastly, to a system of concurrent lines we shall have a corresponding system of coaxal circles on the sphere U.

272. From the last article we see that, being given any graphic property of the conic

$$(b-a)y^2+(c-a)z^2+(d-a)w^2=0,$$

we shall get a corresponding graphic property of the cyclide WU by the following system of substitutions:—

	$(b-a)y^2+(c-a)z^2+(d-a)w^2$.	(WU).
I.	*a*. For a point on,	A. A pair of inverse points.
	b. A point having any permanent relation to,	B. A pair of points having a corresponding relation.
	c. A system of collinear points,	C. A system of concyclic inverse pairs of points.
II.	*a′*. A tangent to,	A′. A generating circle of
	b′. A line having any permanent relation to,	B′. A circle having a corresponding relation to
	c′. A system of concurrent lines.	C′. A system of coaxal circles.

273. If we take the reciprocal of the conic $(b-a)y^2+(c-a)z^2+(d-a)w^2=0$, that is, the conic in tangential coordinates $(b-a)\mu^2+(c-a)\nu^2+(d-a)\varrho^2$, we get properties of WU, by substitutions, reciprocal to the foregoing; hence we are to substitute from the last article, for

$$a,\ b,\ c;\ \mathrm{A}',\ \mathrm{B}',\ \mathrm{C}'.$$
$$a',\ b',\ c';\ \mathrm{A},\ \mathrm{B},\ \mathrm{C}.$$

Cor. 1. Hence, being given any graphic property of a plane conic, we can get two correlative properties of a sphero-quartic.

Cor. 2. The properties of bicircular quartics which are derived by substitutions from those of conics have their analogues in sphero-quartics.

Cor. 3. If two sphero-quartics have one centre of inversion common to both, they

have four common generating circles; for the two conics which lie on the polar plane of the common centre of inversion have four common tangents.

Cor. 4. If $W=a\alpha^2+b\beta^2+c\gamma^2+d\delta^2=0$, $W'=a'\alpha^2+b'\beta^2+c'\gamma^2+d'\delta^2=0$ be two cyclides, the sphero-quartics (WU) and (W'U) have sixteen common generating circles; for they have four centres of inversion common, namely, the centres of the spheres α, β, γ, δ.

CHAPTER XIV.

SECTION I.—*Poles and Polars.*

Observation. All the spheres which we shall have occasion to use in this and the following chapter will be of the form $x\alpha+y\beta+z\gamma+w\delta$, where x, y, z, w are numerical coefficients.

274. If $(a, b, c, d, l, m, n, p, q, r \between \alpha, \beta, \gamma, \delta)^2=0$ be a cyclide, and

$$x_1\alpha+y_1\beta+z_1\gamma+w_1\delta=S_1, \quad x_2\alpha+y_2\beta+z_2\gamma+w_2\delta=S_2$$

be two spheres, then the condition that $\lambda S_1+\mu S_2=0$ may be a generating sphere of the cyclide is given by the determinant

$$\begin{vmatrix} a, & n, & m, & p, & \lambda x_1+\mu x_2, \\ n, & b, & l, & q, & \lambda y_1+\mu y_2, \\ m, & l, & c, & r, & \lambda z_1+\mu z_2, \\ p, & q, & r, & d, & \lambda w_1+\mu w_2, \\ \lambda x_1+\mu x_2, & \lambda y_1+\mu y_2, & \lambda z_1+\mu z_2, & \lambda w_1+\mu w_2, & 0, \end{vmatrix}=0. \qquad (167)$$

This determinant may be written $\lambda^2\Sigma'+2\lambda\mu\phi+\mu^2\Sigma''=0$, and we have a quadratic for determining the ratio $\lambda:\mu$. Now if $\phi=0$ we shall have the two values $\lambda:\mu$ equal, but with contrary signs. Hence $\phi=0$ is the condition that the spheres S_1, S_2 and the two generating spheres of the cyclide whose centres are collinear with their centres, or, in other words, the two generating spheres which are coaxal with them, should form an harmonic system of spheres.

Def. An harmonic system of spheres is a system passing through a common circle and whose centres form an harmonic row of points; this system possesses the property that four tangent planes through any common tangent line form an harmonic system; or again, that the segments which these spheres intercept on the line of collinearity of their centres may be an harmonic system of segments on that axis. See CHASLES, 'Sections Coniques,' art. 136.

275. The equation $\phi=0$ is the determinant

$$\begin{vmatrix} a, & n, & m, & p, & x_1, \\ n, & b, & l, & q, & y_1, \\ m, & l, & c, & r, & z_1, \\ p, & q, & r, & d, & w_1, \\ x_2, & y_2, & z_2, & w_2, & 0, \end{vmatrix}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (168)$$

And if we suppose x_1, y_1, z_1, w_1 constant while x_2, y_2, z_2, w_2 vary, removing the suffixes from the lower row in the determinant, we see that if the centre of a variable sphere $S=0$ moves along the plane

$$\begin{vmatrix} a, & n, & m, & p, & x_1, \\ n, & b, & l, & q, & y_1, \\ m, & l, & c, & r, & z_1, \\ p, & q, & r, & d, & w_1, \\ x, & y, & z, & w, & 0, \end{vmatrix} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (169)$$

then the sphere S, the sphere $S_1=x_1\alpha+y_1\beta+z_1\gamma+w_1\delta=0$, and the coaxal generating spheres form an harmonic pencil of spheres. Now the sphere $S=x\alpha+y\beta+z\gamma+w\delta$, whose centre moves in the plane (169), evidently passes through two fixed points, namely, the two limiting points of the Jacobian sphere U of $\alpha, \beta, \gamma, \delta$, and of the plane (169); I shall call these points the pole points of the sphere $x_1\alpha+y_1\beta+z_1\gamma+w_1\delta$.

Cor. The plane (169) is the polar plane of the centre of S_1 with respect to the focal quadric of the cyclide.

276. If two spheres be such that one of them, A, passes through the pole points of the other, B, then, conversely, B passes through the pole points of A. This is evident from the determinants of the last article, from which it appears that the relation between the spheres is reciprocal. I shall extend the known terms of conics and quadrics, and call two such spheres conjugate spheres, and their two pairs of pole points conjugate pairs of pole points.

277. If two circles in space be such that the pole points of any sphere passing through one lie on the other, then, conversely, the pole points of any sphere passing through the latter lie on the former. This is analogous to the theorem in quadrics, that if two lines A and B be such that the polar plane of any point of A passes through B, then, conversely, the polar plane of any point of B passes through A, and may be derived from it by the substitution explained in the last chapter.

278. If $W=(*\between\alpha, \beta, \gamma, \delta)^2=0$ be a cyclide, and $\alpha', \beta', \gamma', \delta'$ the sphero-coordinates of a pair of inverse points of W, that is, the pair of points given as common to the matrix

$$\begin{Vmatrix} \alpha, & \beta, & \gamma, & \delta, \\ \alpha', & \beta', & \gamma', & \delta', \end{Vmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (170)$$

and $\alpha'', \beta'', \gamma'', \delta''$ the sphero-coordinates of another pair of inverse points, then $\lambda\alpha'+\mu\alpha''$ &c. will be the sphero-coordinates of a pair of points concyclic with $\alpha', \beta', \gamma', \delta'$ and $\alpha'', \beta'', \gamma'', \delta''$; and if these satisfy the equation of W, we shall have

$$\lambda^2W'+2\lambda\mu P+\mu^2W''=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (171)$$

Now if $P=0$, the circle through $\alpha', \beta', \gamma', \delta'$; $\alpha'', \beta'', \gamma'', \delta''$ meets the cyclide in two pairs of inverse points, which are harmonic conjugates to the two pairs $\alpha', \beta', \gamma', \delta'$; $\alpha'', \beta'', \gamma'', \delta''$ (see

CHASLES, 'Sections Coniques,' art. 136, also 'Bicircular Quartics,' arts. 153–155); but P is

$$\equiv\left(\alpha'\frac{d}{d\alpha''}+\beta'\frac{d}{d\beta''}+\gamma'\frac{d}{d\gamma''}+\delta'\frac{d}{d\delta''}\right)W''.$$

Hence, omitting the double accents, we see that the equation of the sphere of which α', β', γ', δ' are the pole points is

$$\left(\alpha'\frac{d}{d\alpha}+\beta'\frac{d}{d\beta}+\gamma'\frac{d}{d\gamma}+\delta'\frac{d}{d\delta}\right)W=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (172)$$

And we have evidently the following theorem:—*If through a pair of inverse points* $(\alpha', \beta', \gamma', \delta')$ *we describe any circle* X *cutting the polar sphere of* $(\alpha', \beta', \gamma', \delta')$ *in a pair of inverse points* $(\alpha'', \beta'', \gamma'', \delta'')$, X *will cut the cyclide in two other pairs of inverse points, such that the four segments made on* X *by the pairs of points* $(\alpha', \beta', \gamma', \delta')$, $(\alpha'', \beta'', \gamma'', \delta'')$ *and by the cyclide are harmonic* (see 'Bicircular Quartics,' art. 153).

279. From the last article we see that, if $(\alpha', \beta', \gamma', \delta')$ be a pair of inverse points on the cyclide, the equation of the generating sphere which touches the cyclide at $(\alpha', \beta', \gamma', \delta')$ is

$$\left(\alpha'\frac{d}{d\alpha}+\beta'\frac{d}{d\beta}+\gamma'\frac{d}{d\gamma}+\delta'\frac{d}{d\delta}\right)W=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (173)$$

This equation establishes a relation between the coordinates $(\alpha', \beta', \gamma', \delta')$ of a pair of inverse points of the cyclide, and $(\alpha, \beta, \gamma, \delta)$ the sphero-coordinates of any other pair of inverse points on the generating sphere which touches the cyclide at $(\alpha', \beta', \gamma', \delta')$; and since the relation is symmetrical with respect to $(\alpha', \beta', \gamma', \delta')$ and $(\alpha, \beta, \gamma, \delta)$, we infer the following theorem:—*If through any pair of inverse points we describe a generating sphere of the cyclide, the locus of all their points of contact is the sphero-quartic which is given as the curve of intersection of* W *with the sphere* (173), *or the polar sphere of* $(\alpha', \beta', \gamma', \delta')$.

280. The discriminant of the equation (171) is

$$W'W''=P^2;$$

and by omitting the double accents we see that the equation of the binodal cyclide which circumscribes W, and which has the pair of points $(\alpha', \beta', \gamma', \delta')$ as nodes, is

$$WW'=\left\{\left(\alpha'\frac{d}{d\alpha}+\beta'\frac{d}{d\beta}+\gamma'\frac{d}{d\gamma}+\delta'\frac{d}{d\delta}\right)W\right\}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (174)$$

281. Since a cyclide has five spheres of inversion, taking any point A, we get five points, namely, the inverses of A with respect to the five spheres of inversion of the cyclide. Let the inverse points be A_1, A_2, A_3, A_4, A_5; and, with the five pairs of inverse points (AA_1), (AA_2), (AA_3), (AA_4), (AA_5), we get by the last article five binodal cyclides circumscribed to W, and these binodals will have one common node, namely, the point A; the other nodes of these circumscribed cyclides will be the five points A_1, A_2, A_3, A_4, A_5.

282. If we invert the cyclide W from the point A, the five binodal cyclides of the last article invert into five cones of the second degree, each having double contact with the inverse cyclide (see art. 187). Now all the points of contact of the five double tangent cones lie on five concentric spheres. Hence we have the following theorem:—

If five binodal cyclides circumscribed to a cyclide W have one common node, their curves of contact with W are five sphero-quartics lying on five spheres having a common radical plane.

283. If $\alpha', \beta', \gamma', \delta'$; $\alpha'', \beta'', \gamma'', \delta''$ be two inverse pairs of points with respect to the sphere of inversion U of the cyclide W, the binodal cyclides which have these pairs of points respectively as nodes, and which circumscribe W, touch W along the sphero-quartics in which it is intersected by the polar spheres of $(\alpha', \beta', \gamma', \delta')$, $(\alpha'', \beta'', \gamma'', \delta'')$; hence the points of contact of generating spheres through both pairs lie on the circle of intersection of the polar spheres: but the plane of this circle intersects W in a bicircular quartic, and the bicircular quartic and the circle intersect in four points; hence there will be four points of contact, and consequently only two generating spheres can be described through two pairs of inverse points.

This theorem may be otherwise stated. Thus, it is plain that the circle through two pairs of inverse points is reciprocal to the circle in which their polar spheres intersect, and then we have the theorem, *that through any circle can be described two generating spheres; their points of contact are concyclic, and lie on the reciprocal circle.*

284. Since when a sphere of inversion U and a focal quadric F are given the cyclide is determined, and if nine points are given the quadric is determined, it hence follows *that, being given a sphere* U *and nine spheres which are orthogonal to it, a cyclide can be described having* U *for a sphere of inversion, and the nine spheres as generating spheres.*

285. Since, being given any eight points, three quadrics can be described to touch a given plane, we have the theorem, *that, being given any eight generating spheres of a cyclide, three cyclides can be described through the same pair of inverse points with respect to* U (see art. 268, 2°), *and the cyclides are mutually orthogonal* (see art. 119).

286. If two quadrics intersect in the same eight points, all quadrics passing through these eight points have a common curve of intersection. Hence, *if two cyclides* W, W′ *have eight generating spheres common, every cyclide having these eight spheres as generators will have also as generators all the generators common to* W, W′. We shall in the next chapter find the equation of the surface formed by all the generators common to two cyclides, and also give some of its properties.

287. Given seven points or tangent planes common to a series of quadrics, then an eighth point or tangent plane common to the system is determined. Hence, *being given seven generating spheres or pairs of inverse points common to a system of cyclides, then an eighth generating sphere or pair of inverse points common to the whole system is determined.*

288. *If a system of cyclides pass through the same eight pairs of inverse points, their polar spheres with respect to a given pair of inverse points have a common radical plane.*

For if P and Q be the polar spheres of a given pair of inverse points with respect to W and W′, then $P+\lambda Q$ is the polar sphere of the same pair of inverse points with respect to $W+\lambda W'$.

289. By reciprocating the theorem of the last article we get the theorem :—*If a system*

of cyclides have eight common generating spheres, the locus of the pole points of a fixed sphere is a circle.

290. *If a system of cyclides pass through the same eight pairs of inverse points, that is, if they have a common curve of intersection, the polar circles of a fixed circle generate a cyclide.*

Let the polar spheres of two fixed pairs of inverse points be $P+\lambda Q$ and $P'+\lambda Q'$; eliminating λ, we get the cyclide $PQ'-P'Q=0$.

291. *Reciprocally, if a system have eight common generating spheres, the polar circles of a fixed circle generate a cyclide.*

292. *If a system of cyclides pass through a common curve, the locus of the pole points of a fixed sphere is a torse curve of the sixth degree.*

Demonstration. Let the polar spheres of three pairs of inverse points lying in the fixed sphere be $P+\lambda Q$, $P'+\lambda Q'$, $P''+\lambda Q''$; then, eliminating λ, we get the system of determinants

$$\left\| \begin{matrix} P, & P', & P'', \\ Q, & Q', & Q'', \end{matrix} \right\| \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (175)$$

which represents a twisted curve of the sixth degree. For the intersection of the cyclides $PQ'-P'Q$, $PQ''-P''Q$, each of which has the imaginary circle at infinity as a double line, is a twisted curve of the eighth degree; but this includes the circle (PQ), which is not part of the intersection of the cyclides $PQ''-P''Q$, $P'Q''-P''Q'$; there is, therefore, only a curve of the sixth degree common to the three determinants of the matrix (175).

Cor. The cone whose vertex is the centre of U, the common sphere of inversion of the cyclides, and which stands on the curve of the sixth degree, is only of the third degree. For any plane through the vertex of the cone meets the curve in six points; but these are inverse two by two, since the curve is evidently an anallagmatic, and therefore only three edges of the cone lie in the plane.

293. *Given seven pairs of inverse points of a cyclide, the polar spheres of a given inverse pair of points pass through a given pair of inverse points.*

For evidently the polar sphere of a given fixed pair of inverse points with respect to $W+\lambda W'+\mu W''$ will be of the form $P+\lambda P'+\mu P''$, and will therefore pass through a given fixed pair of inverse points, namely, the two points common to the spheres P, P', P''.

Reciprocally, given seven generating spheres of a cyclide, the locus of the pole points of a fixed sphere is a fixed sphere.

294. If $W=(*\between\alpha, \beta, \gamma, \delta)^2=0$ be a given cyclide, we have seen that $(*\between\lambda, \mu, \nu, \varrho)^2=0$ is the tangential equation of the focal quadric; but if the discriminant vanish of the equation of a quadric in tangential coordinates, it represents a conic in space, and the corresponding cyclide will be binodal. Hence we have the theorem, *if the discriminant vanish of the equation of a cyclide, the cyclide will be a binodal cyclide.*

295. Since the discriminant contains the coefficients in the fourth degree, it follows

that we have a biquadratic to solve to determine λ, in order that $W+\lambda W'$ may represent a binodal cyclide. Hence, through the curve of intersection of two cyclides, four binodal cyclides may be described. The binodes of these binodals are thus determined. If we denote by W_1, W_2, W_3, W_4 the differentials of W with respect to α, β, γ, δ respectively, and by λ_1, λ_2, λ_3, λ_4 the four roots of the biquadratic in λ, then any three of the four spheres

$$W_1+\lambda_1 W_1', \quad W_2+\lambda_1 W_2', \quad W_3+\lambda_1 W_3', \quad W_4+\lambda_1 W_4'$$

will determine by their mutual intersections the binodes of one of the binodals; and the binodes of the remaining binodals are got from these by using the remaining roots of the biquadratic in place of λ_1, namely, λ_2, λ_3, λ_4 respectively.

296. *There are four spheres whose pole points are the same with respect to all the cyclides passing through a common curve of intersection of two cyclides, namely, the polar spheres of the four pairs of nodes of the four binodals of the last article.* For to express the condition that

$$\alpha W_1'+\beta W_2'+\gamma W_3'+\delta W_4',$$
$$\alpha X_1' +\beta X_2' +\gamma X_3' +\delta X_4'$$

should represent the same spheres, we find the following set of determinants:

$$\left\| \begin{matrix} W_1', & W_2', & W_3', & W_4', \\ X_1', & X_2', & X_3', & X_4', \end{matrix} \right\|$$

and these satisfy by the last article the binodes of the binodal cyclides.

297. *The four spheres are such that the two points common to any three are the pole points of the fourth; and, conversely, the four pairs of binodal points are such that the sphere determined by any three is the polar sphere of the fourth pair.* Thus, if the two cyclides be W and W', their equations in terms of the four spheres will be of the form

$$aX^2+bY^2+cZ^2+dV^2,$$
$$a'X^2+b'Y^2+c'Z^2+d'V^2,$$

and the nodes of the binodal cyclides are the four pairs of points

$$(XYZ), \quad (XYV), \quad (XZV), \quad (YZV).$$

It is to be remembered that the spheres X, Y, Z, V are not mutually orthogonal.

298. If the cyclide W' break up into two spheres, the form $W+\lambda W'$ becomes $W+\lambda LM$. In general the intersection of two cyclides is a twisted curve of the eighth degree; but if one of the two cyclides reduce to two spheres, the intersection becomes two sphero-quartics. Any pair of inverse points on the circle LM has the same polar sphere with respect to all the cyclides of the system $W+\lambda LM$; and in particular all the cyclides of the system have the same generating spheres at the four points where W is met by the circle LM.

Lastly, all the cyclides of the system are enveloped by four binodal cyclides. For if

the four common generating spheres be α, β, γ, δ, then the nodes of the four enveloping binodals are the four pairs of inverse points (α, β, γ), (α, γ, δ), (α, β, δ), (β, γ, δ).

299. If a sphero-quartic be common to three cyclides, each pair must have another sphero-quartic, and the spheres through these sphero-quartics are coaxal.

300. W, W′ are two cyclides having a common sphere of inversion U; it is required to find the locus of the pole points of the generating spheres of W′ with respect to W.

Let W, W′ be reduced to their canonical forms,

$$W \equiv a\alpha^2+b\beta^2+c\gamma^2+d\delta^2, \quad W' \equiv a'\alpha^2+b'\beta^2+c'\gamma^2+d'\delta^2.$$

Now let $(\alpha', \beta', \gamma', \delta')$ be a pair of inverse points with respect to U, then the polar sphere of $(\alpha', \beta', \gamma', \delta')$ with respect to W is

$$a\alpha'\alpha+b\beta'\beta+c\gamma'\gamma+d\delta'\delta=0;$$

and the condition that this should be a generating sphere of W′ is

$$\frac{a^2\alpha'^2}{a'}+\frac{b^2\beta'^2}{b'}+\frac{c^2\gamma'^2}{c'}+\frac{d^2\delta'^2}{d'}=0;$$

and omitting the accents, we have the locus required,

$$\left(\frac{a^2}{a'}\right)\alpha^2+\left(\frac{b^2}{b'}\right)\beta^2+\left(\frac{c^2}{c'}\right)\gamma^2+\left(\frac{d^2}{d'}\right)\delta^2=0. \quad \ldots\ldots \quad (176)$$

301. If we denote the cyclide (176) by $W''=0$, we see that the equations of the focal quadrics of W, W′, W″, in tangential coordinates, are

$$a\lambda^2+b\mu^2+c\nu^2+d\varrho^2=0,$$
$$a'\lambda^2+b'\mu^2+c'\nu^2+d'\varrho^2=0,$$
$$\frac{a^2}{a'}\lambda^2+\frac{b^2}{b'}\mu^2+\frac{c^2}{c'}\nu^2+\frac{d^2}{d'}\varrho^2=0;$$

and the third is the reciprocal of the second with respect to the first. Hence we have the following theorem:—*If* W, W′, W″ *be three cyclides having a common sphere of inversion* U, *and* F, F′, F″ *be the focal quadrics of* W, W′, W″ *corresponding to* U, *then if* F″ *be the reciprocal of* F′ *with respect to* F, W″ *will be the reciprocal of* W′ *with respect to* W.

302. Since the reciprocal of a circle in space with respect to a cyclide is another circle in space, hence, if a variable circle move along three circles, its reciprocal will move along three circles reciprocal to the former; *so that the reciprocal of a cyclide described by the motion of a variable circle is another cyclide described by the motion of another variable circle.* This corresponds to the theorem that the reciprocal of a ruled surface is a ruled surface.

303. *If a pair of inverse points move along a fixed sphere, the locus of the pair of inverse points common to their polar spheres, with respect to three cyclides having a common sphere of inversion* U, *is a cyclide of the sixth degree, having* U *for a sphere of inversion.*

Demonstration. If the polar spheres of the three cyclides with respect to the three pairs of inverse points $(\alpha', \beta', \gamma', \delta')$, $(\alpha'', \beta'', \gamma'', \delta'')$, $(\alpha''', \beta''', \gamma''', \delta''')$ be $X', Y', Z', X'', Y'', Z'', X''', Y''', Z'''$, then the polar spheres of the cyclides with respect to the pair of inverse points whose sphero-coordinates are $k\alpha'+l\alpha''+m\alpha'''$, $k\beta'+l\beta''+m\beta''$, $k\gamma'+l\gamma''+m\gamma'''$, $k\delta'+l\delta''+m\delta''$ will plainly be $kX'+lX''+mX'''$, $kY'+lY''+mY'''$, $kZ'+lZ''+mZ'''$. Hence, eliminating k, l, m, we get the required locus,

$$\begin{vmatrix} X', & X'', & X''', \\ Y', & Y'', & Y''', \\ Z', & Z'', & Z''', \end{vmatrix} = 0, \qquad (177)$$

a cyclide of the sixth degree having the circle at infinity as a triple line.

Cor. *If the pair of inverse points move along a circle, the locus of the intersection of their polar spheres with respect to two cyclides will be a cyclide of the fourth degree.*

304. From article 301, we can easily infer theorems in cyclide reciprocation from known theorems in quadric reciprocation. Thus, if two spheres be concentric, the reciprocal of one with respect to the other is a concentric sphere. Hence, *if two Cartesian cyclides having a common sphere of inversion have a common triple focus, the reciprocal of one with respect to the other is a Cartesian cyclide having the same triple focus;* or, since the theorem concerning the spheres may be enunciated thus, if tangent planes to a sphere intersect at given angles, the locus of their point of intersection is a concentric sphere, and the envelope of the plane through their points of contact is another concentric sphere. Hence we infer that *if three generating spheres of a Cartesian cyclide intersect at given angles, the locus of their points of intersection is a Cartesian cyclide having the same triple focus, and the envelope of the sphere through their six points of contact is another Cartesian cyclide having also the same triple focus.* These theorems may evidently be inferred by the methods of substitution given in the last Chapter.

305. If we reciprocate one sphere with respect to another not concentric, we get a quadric of revolution. Hence *the reciprocal of a Cartesian cyclide with respect to another Cartesian cyclide having a different triple focus is a symmetrical cyclide, that is, a cyclide having one of its spheres of inversion opened out into a plane, the corresponding focal quadric being one of revolution.*

306. If we reciprocate a surface of revolution with respect to a sphere, we get a general quadric. Hence, *if we reciprocate a cyclide having a plane of symmetry with respect to a Cartesian cyclide, we get a general cyclide.*

307. The principles explained in recent articles will obviously give some of the systems of substitutions explained in the last Chapter; and, conversely, the results of this Chapter may be derived from the substitutions of the last. It is unnecessary to pursue the subject further, and I shall conclude the section with the two following theorems:—

1°. The locus of the intersection of three rectangular tangent planes to a quadric is a sphere. Hence *the locus of the pairs of points common to three generating spheres of a cyclide which are mutually orthogonal is a Cartesian cyclide.*

2°. Every central quadric has two systems of circular sections. Hence *every quartic cyclide has two systems circumscribed to it of binodal Cartesian cyclides, and the locus of their nodes is two right lines respectively perpendicular to the directions of the planes of circular sections of the focal quadric.*

SECTION II.—*Poles and Polars. Sphero-quartics.*

308. The investigation of the polar properties of sphero-quartics is analogous to that employed in the last section for cyclides.

Thus if $(a, b, c, f, g, h)(\alpha, \beta, \gamma)^2=0$, where α, β, γ are circles on a sphere U, be the equation of a sphero-quartic, and if $\lambda_1\alpha+\mu_1\beta+\nu_1\gamma=C_1$ and $\lambda_2\alpha+\mu_2\beta+\nu_2\gamma=C_2$ be two circles, then the condition that $lC_1+mC_2=0$ should be a generating circle is given by the determinant

$$\begin{vmatrix} a, & h, & g, & l\lambda_1+m\lambda_2, \\ h, & b, & f, & l\mu_1+m\mu_2, \\ g, & f, & c, & l\nu_1+m\nu_2, \\ l_1\lambda+m\lambda_2, & l\mu_1+m\mu_2, & l\nu_1+m\nu_2, & 0. \end{vmatrix} \quad . \quad . \quad . \quad . \quad (178)$$

This determinant may be written in the form

$$l^2\Sigma'+2lm\phi+m^2\Sigma''=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (179)$$

Hence $\phi=0$ is the condition that the circles C_1, C_2, and the two generating circles whose centres lie on the same great circle with their centres, should form an harmonic pencil of circles, or it is the condition that their centres should form an harmonic row of points; or, again, it is the condition that their diameters should form an harmonic system of segments on the same great circle of U.

309. The equation $Q=0$ is the determinant

$$\begin{vmatrix} a, & h, & g, & \lambda_1, \\ h, & b, & f, & \mu_1, \\ g, & f, & c, & \nu_1, \\ \lambda_2 & \mu_2 & \nu_2, & 0, \end{vmatrix}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (180)$$

This is the condition that the circles C_1 and C_2 may be conjugate circles with respect to the quartic; if the suffixes be removed from the lower row, we see that, if the centre of a variable circle $C\equiv\lambda\alpha+\mu\beta+\nu\gamma=0$ move along the great circle

$$\begin{vmatrix} a, & h, & g, & \lambda_1, \\ h, & b, & f, & \mu_1, \\ g, & f, & c, & \nu_1, \\ \lambda, & \mu, & \nu, & 0, \end{vmatrix}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (181)$$

then $C=0$, $C_1=\lambda_1\alpha+\mu_1\beta+\nu_1\gamma$, and the two generating circles whose centres are on the same great circle with their centres form an harmonic pencil; but if a variable circle

whose centre moves along a great circle cuts a given circle J orthogonally, it will pass through two fixed points; these fixed points are the limiting points of J and the great circle; or if we denote, as Dr. SALMON does, the equation of a great circle by an equation of the first degree in $x\ y\ z$, say $L=0$, and the circle J by the equation $S^2-M=0$, then the two limiting points will be given as those for which the discriminant of (S^2-M+kL) vanishes. These points will be the pole points of the circle $\lambda_1\alpha+\mu_1\beta+\nu_1\gamma=0$ with respect to the quartic $(a, b, c, f, g, h \between \alpha, \beta, \gamma)^2=0$.

Cor. 1. The great circle (181) is the polar of the centre of $C_1=\lambda_1\alpha+\mu_1\beta+\nu_1\gamma$ with respect to the sphero-conic whose tangential equation is

$$A\lambda^2+B\mu^2+C\nu^2+2H\lambda\mu+2F\mu\nu+2G\nu\lambda=0,$$

where, as usual, $A=bc-f^2$, $B=ca-g^2$, &c.

Cor. 2. If two circles be such that one of them, A, passes through the pole points of B, then, conversely, B passes through the pole points of A.

310. If $(\alpha', \beta', \gamma')$ be the cyclic coordinates of a pair of inverse points, that is, the pair of points given by the system of circles

$$\left\| \begin{matrix} \alpha, & \beta, & \gamma, \\ \alpha', & \beta', & \gamma', \end{matrix} \right\|$$

and $\alpha'', \beta'', \gamma''$ the cyclic coordinates of another pair of points, then $l\alpha'+m\alpha''$, $l\beta'+m\beta''$, $l\gamma'+m\gamma''$ will be the coordinates of a pair of points concyclic with them; and if these satisfy the equation of the sphero-quartic, which we may denote by Q, we shall have

$$l^2Q'+2lmP+m^2Q''=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (182)$$

Now, if $P=0$ the circle through $(\alpha', \beta', \gamma')$, $(\alpha'', \beta'', \gamma'')$ meets the quartic in two pairs of points which are harmonic conjugates with respect to the two pairs $(\alpha', \beta', \gamma')$, $(\alpha'', \beta'', \gamma'')$; but P is

$$\left(\alpha'\frac{d}{d\alpha''}+\beta'\frac{d}{d\beta''}+\gamma'\frac{d}{d\gamma''}\right)Q''.$$

Hence the equation of the polar circle of the points $(\alpha', \beta', \gamma')$ is

$$\left(\alpha'\frac{d}{d\alpha}+\beta'\frac{d}{d\beta}+\gamma'\frac{d}{d\gamma}\right)Q=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (182\,a)$$

Cor. 1. From this article we have evidently the following theorem:—If through a pair of inverse points α', β', γ' we describe any circle Z cutting the polar circle of $(\alpha', \beta', \gamma')$ in a pair of inverse points $(\alpha'', \beta'', \gamma'')$, Z will cut the quartic in two other pairs of points, such that the four segments made on Z by $(\alpha', \beta', \gamma')$, $(\alpha'', \beta'', \gamma'')$ and by the quartic are harmonic.

Cor. 2. If $(\alpha', \beta', \gamma')$ be a pair of points, the generating circle which touches Q at $(\alpha', \beta', \gamma')$ is

$$\left(\alpha'\frac{d}{d\alpha}+\beta'\frac{d}{d\beta}+\gamma'\frac{d}{d\gamma}\right)Q=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (183)$$

Cor. 3. If through a pair of inverse points we describe two generating circles of the

quartic, their points of contact with the quartic lie on the polar circle of the given pair of inverse points.

311. The discriminant of the equation (182) is

$$Q'Q''=P^2.$$

Hence the equation of the pair of generating circles of the sphero-quartic which pass through $(\alpha', \beta', \gamma')$ is

$$QQ'=\left\{\left(\alpha'\frac{d}{d\alpha}+\beta'\frac{d}{d\beta}+\gamma'\frac{d}{d\gamma}\right)Q\right\}^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (184)$$

This equation can also, as is evident, be written in the form

$$\begin{vmatrix} a, & h, & g, & \beta'\gamma-\beta\gamma', \\ h, & b, & f, & \gamma'\alpha-\gamma\alpha', \\ g, & f, & c, & \alpha'\beta-\alpha'\beta, \\ \beta'\gamma-\beta\gamma', & \gamma'\alpha-\gamma\alpha', & \alpha'\beta-\alpha\beta' & 0, \end{vmatrix}=0. \quad . \quad . \quad . \quad (185)$$

In like manner the equation of the four points in which the circle $\lambda_1\alpha+\mu_1\beta+\nu_1\gamma$ cuts the quartic is

$$\Sigma'\,\Sigma''-\phi^2=0 \text{ (see art. 308)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (186)$$

which may also be written in the form

$$\begin{vmatrix} A, & H, & G, & \mu_1\nu-\mu\nu_1, \\ H, & B, & F, & \nu_1\lambda-\nu\lambda_1, \\ G, & F, & C, & \lambda_1\mu-\lambda\mu_1, \\ \mu_1\nu-\mu\nu_1, & \nu_1\lambda-\nu\lambda_1, & \lambda_1\mu-\lambda\mu_1 & 0. \end{vmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (187)$$

CHAPTER XV.

Invariants and Covariants of Cyclides.

312. It is always possible in an infinity of ways to choose four spheres $\alpha, \beta, \gamma, \delta$ so that the equations of two cyclides having a common sphere of inversion can be thrown into the forms

$$W=(a, b, c, d, l, m, n, p, q, r\between\alpha, \beta, \gamma, \delta)^2=0,$$

$$W'=(a', b', c', d', l', m', n', p', q', r'\between\alpha, \beta, \gamma, \delta)^2=0.$$

For each of these equations contains explicitly nine constants, and each of the spheres $\alpha, \beta, \gamma, \delta$ contains implicitly four constants, so that we have thirty-four constants at our disposal, and we require but twenty-two. For the two cyclides are determined when the common sphere of inversion and the two focal quadrics are given; hence the number of constants required is $4+9\times2=22$.

313. Denoting the discriminants of W, W′ by Δ, Δ', we have Δ, Δ' given by the determinants

$$\Delta=\begin{vmatrix} a, & n, & m, & p, \\ n, & b, & l\,, & q, \\ m, & l, & c\,, & r, \\ p\,, & q, & r\,, & d, \end{vmatrix}, \qquad \Delta=\begin{vmatrix} a'\,, & n', & m', & p', \\ n', & b', & l'\,, & q', \\ m', & l', & c'\,, & r', \\ p', & q', & r'\,, & d'. \end{vmatrix} \quad . \;\; . \;\; . \quad (188)$$

Then the discriminant of $k\mathrm{W}+\mathrm{W}'$ will be got from Δ by writing in place of a, b, &c. $ka+a'$, $kb+b'$, &c.; the result will be a quartic in k, which I shall call the invariant equation of the two cyclides, and write in the form

$$k^4\Delta+k^3\Theta+k^2\Phi+k\Theta'+\Delta'=0. \quad . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \quad (189)$$

Since there are four values of k which satisfy this equation, we see that through the curve (WW′) can be drawn four binodal cyclides, that is, four cyclides each having two conic nodes. If we eliminate k between $k\mathrm{W}+\mathrm{W}'$ and (189), we shall get the equation of these four binodal cyclides, namely,

$$\Delta'\mathrm{W}^4-\Theta'\mathrm{W}^3\mathrm{W}'+\Phi\mathrm{W}^2\mathrm{W}'^2-\Theta\mathrm{W}\mathrm{W}'^3+\Delta\mathrm{W}'^4=0. \quad . \;\; . \;\; . \quad (190)$$

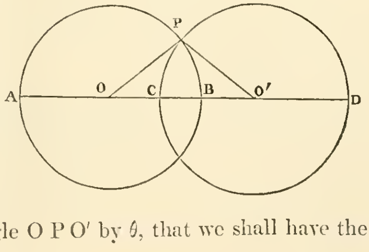
314. Since the equations of W, W′ are the same in form as the tangential equation of their focal quadrics F, F′, and if F, F′ touch, W, W′ will have double contact, hence it follows that the condition of W, W′ having double contact is the vanishing of the discriminant of the invariant equation (189); ∴ the tact-invariant of W, W′ is

$$4(12\Delta\Delta'-3\Theta\Theta'+\Phi^2)^3-(72\Delta\Delta'\Phi+9\Theta\Theta'\Phi-27\Delta\Theta'^2-27\Delta'\Theta^2-2\Phi^3)^2. \quad . \quad (191)$$

315. The tact-invariants of two conics and two quadrics are the analytical expression of remarkable geometrical properties which have not been hitherto noticed by any writer so far as I am aware; on this account, and because extensions of them hold for the tact-invariants of two bicircular quartics and two cyclides, I shall give their investigations here, and we shall incidentally find results that are important independently of the properties that we have alluded to, and which we now proceed to demonstrate.

316. If A, B, C, D be four points ranged in alternate order on a right line, the six anharmonic ratios of A, B, C, D can be expressed in a way that bears a remarkable analogy to the six trigonometrical functions of an angle.

Fig. 8.



On A B and C D describe circles; let O, O′ be their centres, P one of their points of intersection, then the angle O P O′ equal angle of intersection of the circles; and taking the six anharmonic ratios of A, B, C, D, as given in TOWNSEND'S 'Modern Geometry,' or CHASLES'S 'Géométrie Supérieure,' it is easy to see, if we denote the angle O P O′ by θ, that we shall have the equations:

$$
\begin{array}{ll}
(1) & \mathrm{CA}\,.\,\mathrm{BD} : \mathrm{BA}\,.\,\mathrm{CD} = \sin^2 \tfrac{1}{2}\theta, \\
(2) & \mathrm{CB}\,.\,\mathrm{AD} : \mathrm{AB}\,.\,\mathrm{CD} = \cos^2 \tfrac{1}{2}\theta, \\
(3) & \mathrm{AC}\,.\,\mathrm{BD} : \mathrm{BC}\,.\,\mathrm{AD} = -\tan^2 \tfrac{1}{2}\theta, \\
(4) & \mathrm{BA}\,.\,\mathrm{CD} : \mathrm{CA}\,.\,\mathrm{BD} = \operatorname{cosec}^2 \tfrac{1}{2}\theta, \\
(5) & \mathrm{AB}\,.\,\mathrm{CD} : \mathrm{CB}\,.\,\mathrm{AD} = \sec^2 \tfrac{1}{2}\theta, \\
(6) & \mathrm{BC}\,.\,\mathrm{AD} : \mathrm{AC}\,.\,\mathrm{BD} = -\cot^2 \tfrac{1}{2}\theta.
\end{array}
$$

317. Let there be given two conics referred to their common self-conjugate triangle $S \equiv x^2+y^2+z^2$, $S' \equiv ax^2+by^2+cz^2$, and let us denote by θ', θ'', θ''' the angles (see last article) of the anharmonic ratios of the three quartets of points in which the sides of the self-conjugate triangle is intersected by the two conics. Then for determining θ' we must find the anharmonic ratio in which the side $x=0$ is intersected by the two conics; for that purpose we have the pencil formed by the two pairs of lines $y^2+z^2=0$ and $by^2+cz^2=0$, and we easily get

$$\sin^2 \tfrac{1}{2}\theta' = -\frac{(b^{\frac{1}{2}}-c^{\frac{1}{2}})^2}{4b^{\frac{1}{2}}c^{\frac{1}{2}}},$$

$$\cos^2 \tfrac{1}{2}\theta' = \frac{(b^{\frac{1}{2}}+c^{\frac{1}{2}})^2}{4b^{\frac{1}{2}}c^{\frac{1}{2}}}.$$

Hence

$$\sin^2 \theta' = -\frac{(b-c)^2}{4bc}.$$

Now if we form the invariant equation in k for the two conics S, S′, that is, if we form the discriminant of $kS+S'$, and denote its roots by k', k'', k''', these roots are known to be $-a$, $-b$, $-c$. Hence we have the following system of equations:—

$$\left.\begin{aligned}
\sin^2\theta' &= -(k''-k''')^2 : 4k''k''', \\
\sin^2\theta'' &= -(k'''-k')^2 : 4k'''k', \\
\sin^2\theta''' &= -(k'-k'')^2 : 4k'k''.
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (192)$$

Hence the discriminant for the invariant equation of the two conics S, S′ is $\frac{-64\Delta'^2}{\Delta^2}(\sin^2\theta' . \sin^2\theta'' . \sin^2\theta''')$, or omitting the multiplier $\frac{-64\Delta'^2}{\Delta^2}$, which is numerical, the discriminant is $\sin^2\theta' . \sin^2\theta'' . \sin^2\theta'''$; and as each sine squared is the product of two anharmonic ratios (see art. 316), we have the following theorem, which is the one referred to in art. 315:—

The tact-invariant of two conics is the product of six anharmonic ratios, and the vanishing of any one of these six ratios is a condition of contact of the two conics.

Cor. From the values given for the invariant angles θ', θ'', θ''' in this article, we get

$$e^{2\theta'\sqrt{-1}} = k'' : k''', \quad e^{2\theta''\sqrt{-1}} = k''' : k', \quad e^{2\theta'''\sqrt{-1}} = k' : k''.$$

Hence $\theta'+\theta''+\theta'''=0$, that is, *the sum of the three invariant angles of two conics is equal to zero.*

318. If we take the original conics S, S′, and form the reciprocal of S with respect to S′, we get $a^2x^2+b^2y^2+c^2z^2$; if we denote this by S″, and form the invariant angles of S, S″, we find them to be $2\theta'$, $2\theta''$, $2\theta'''$; similarly, if S‴ be the reciprocal of S′ with respect to S″, the invariant angles of S, S‴ are $3\theta'$, $3\theta''$, $3\theta'''$, and so on. Again, if we denote by $S_{\frac{1}{2}}$ the conic which reciprocates S into S′, the invariant angles of S, $S_{\frac{1}{2}}$ are $\frac{1}{2}\theta'$, $\frac{1}{2}\theta''$, $\frac{1}{2}\theta'''$, &c.

Cor. *If two conics* S, S′ *be so related that a triangle circumscribed to* S *will be inscribed in* S′, *and if we reciprocate* S *with respect to* S′, *the reciprocal conic* S″ *will be related to* S *by the condition that any triangle inscribed in* S″ *will be self-conjugate with respect to* S.

319. From the values given (art. 317), we get $\cos\theta'=\frac{b+c}{2\sqrt{bc}}$; and since $-a=k'$, we have $\cos\theta'\div\sqrt{k'}=(b+c):2\sqrt{-abc}$, with similar values for $\cos\theta''\div\sqrt{k''}$, $\cos\theta'''\div\sqrt{k'''}$. Hence we may write the equation of the conic ϕ, which is the envelope of a line cut harmonically by S, S′ (see SALMON's 'Conics,' p. 334), in the following manner:—

$$\left(\frac{\cos\theta'}{\sqrt{k'}}\right)\lambda^2+\left(\frac{\cos\theta''}{\sqrt{k''}}\right)\mu^2+\left(\frac{\cos\theta'''}{\sqrt{k'''}}\right)\nu^2=0. \quad . \quad . \quad . \quad . \quad . \quad (193)$$

This equation is altogether metrical, having no reference to any particular system of axes, being in fact true for any system whatever of trilinear axes.

Cor. 1. In like manner the equation of SALMON's conic F, which is the locus of points whence tangents to S, S′ form an harmonic pencil, may be written in the form

$$\sqrt{k'}\cos\theta' x^2+\sqrt{k''}\cos\theta'' y^2+\sqrt{k'''}\cos\theta''' z^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (194)$$

Cor. 2. The discriminants of the covariant conics ϕ, F are the quotient and product of the expressions $\cos\theta'.\cos\theta''.\cos\theta'''$ and $\sqrt{k'.k''.k'''}$.

Cor. 3. The reciprocal of S′ with respect to F, that is, with respect to the conic (194), is

$$\cos^2\theta' x^2+\cos^2\theta'' y^2+\cos^2\theta''' z^2=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (195)$$

320. It is easy now to extend the results we have arrived at to the case of two quadrics. Let them be

$$U\equiv ax^2+by^2+cz^2+dw^2=0,$$
$$V\equiv x^2+y^2+z^2+w^2=0;$$

and if the angles be determined thus,

$$e^{2\theta'\sqrt{-1}}=k'':k''', \qquad e^{2\theta(\text{IV})\sqrt{-1}}=k':k^{(\text{IV})},$$
$$e^{2\theta''\sqrt{-1}}=k''':k', \qquad e^{2\theta(\text{V})\sqrt{-1}}=k'':k^{(\text{IV})},$$
$$e^{2\theta'''\sqrt{-1}}=k':k'', \qquad e^{2\theta(\text{VI})\sqrt{-1}}=k''':k^{(\text{IV})},$$

the following Table gives the angles for the pairs of conics in which the faces of the self-conjugate tetrahedron intersect the quadrics:—

Faces.	Angles.		
x,	θ',	θ'',	$\theta^{(\text{vi})}$,
y,	θ''',	$\theta^{(\text{iv})}$,	$\theta^{(\text{vi})}$,
z,	θ''',	$\theta^{(\text{iv})}$,	$\theta^{(\text{v})}$,
w,	θ',	θ'',	θ''';

and then the discriminant of the invariant equation of the two quadrics is

$$\sin^2\theta' . \sin^2\theta'' . \sin^2\theta''' . \sin^2\theta^{(\text{iv})} . \sin^2\theta^{(\text{v})} . \sin^2\theta^{(\text{vi})},$$

which, as in art. 317, is the product of twelve anharmonic ratios. Hence *the tact-invariant of two quadrics is the product of twelve anharmonic ratios, and the vanishing of any one of these ratios is the condition of contact of the two quadrics.*

Cor. *It follows from art.* 317 *that the condition of double contact of two bicircular or two sphero-quartics is expressible as the product of six anharmonic ratios, and, from the present article, of twelve anharmonic ratios, for the double contact of two cyclides.*

321. We now return from this digression (articles 315–320). If the cyclide W′ (see art. 313) be a binodal cyclide, we have $\Delta'=0$; and we proceed to examine the meaning in this case of Θ, Φ, Θ'. Let us take the nodes of W′ as the points common to three of the spheres of reference α, β, γ, then in the equation of W′ (see art. 313) p', q', r', d' all vanish, and we get $\Theta'=d(a'b'c'+2l'm'n'-a'l'^2-b'm'^2-c'n'^2)$, or Θ' vanishes if W′ break up into two spheres, or if the nodes of W′ be on the surface of W. Let the binodal cyclide which circumscribes W, and whose nodes coincide with those of W′, viz.

$$d(a\alpha^2+b\beta^2+c\gamma^2+2l\beta\gamma+2m\gamma\alpha+2n\alpha\beta)-(p\alpha+q\beta+r\gamma)^2=0,$$

be written

$$a''\alpha^2+b''\beta^2+c''\gamma^2+2l''\beta\gamma+2m''\gamma\alpha+2n''\alpha\beta=0,$$

then Φ may be written

$$a''(b'c'-l'^2)+b''(c'a'-m'^2)+c''(a'b'-n'^2)+2l''(m'n'-a'l')+2m''(n'l'-b'm')+2n''(l'm'-c'n'). \quad (196)$$

Hence, by the theory of bicircular quartics (art. 174), Φ *vanishes when the intersections of three harmonic spheres of* W′ *are three circles having double contact with* W. In like manner

$$\left.\begin{aligned} d\Theta=a'(b''c''-l''^2)+b'(c''a''-m''^2)+c'(a''b''-n''^2)+2l'(m''n''-a''l'')\\ +2m'(n''l''-b''m'')+2n'(l''m''-c''n''), \end{aligned}\right\} \quad . \quad (197)$$

or Θ vanishes when the generators of W are harmonic spheres of W′ (see 'Bicircular Quartics,' art. 218).

When W′ breaks up into two spheres, both Δ' and Θ' vanish. Let the two spheres be α, β, then W′ reduces to $n\alpha\beta$, and Φ reduces to $n'^2(r^2-cd)$, or Φ will vanish when the intersection of the two spheres is a circle having double contact with W. In like manner Θ vanishes when the two spheres are conjugate spheres with respect to W. The condition will be satisfied, $\Theta^2=4\Delta\Phi$, if either of the two spheres be a generating sphere of W.

322. Given nine cyclides, W_1, W_2 .... W_9, it is possible in an infinity of ways to deter-

mine nine constants $l_1, l_2, \ldots l_9$ so that $l_1W_1+l_2W_2\ldots+l_9W_9$ may be a perfect square L^2, or the product of two spheres, M and N; it is required to prove that the envelope of the sphere L is a cyclide, and that M and N are conjugate spheres with respect to it.

Demonstration. We can determine a cyclide $W=(a, b\ldots)(\alpha, \beta, \gamma, \delta)^2$ so that the invariant Θ shall vanish for W and each of the nine cyclides, since we have nine equations of the form

$$Aa_1+Bb_1+Cc_1+Dd_1+2Ll_1+2Mm_1+2Nn_1+2Pp_1+2Qq_1+2Rr_1=0, \quad . \quad . \quad (198)$$

A, B, C, &c. being the minors of the determinant Δ (see art. 313), and a_1, b_1, c_1, &c. the coefficients of W_1; hence the mutual ratios of A, B, C are determined. Now if we have separately nine equations of the form (198), we have plainly also

$$A(l_1a_1+l_2a_2\ldots l_9a_9)+\text{\&c.}=0,$$

that is, Θ vanishes for W and every cyclide of the system $l_1W_1+l_2W_2\ldots l_9W_9$. Hence the theorem is proved.

Cor. If the sphere M be given, N passes through a given pair of inverse points, namely, the pole points of M with respect to W.

323. If we are given only eight cyclides, $W_1, W_2\ldots W_8$, and seek to determine the cyclide W as in art. 322, so that the invariant Θ shall vanish for W and each of the eight cyclides, then, since we have only eight conditions, one of the tangential coefficients A, &c. remain undetermined; but we can determine all the rest in terms of that one, so that the tangential equation of W is $\Omega+K\Omega'=0$. Hence the focal quadric of W contains an indeterminate constant in the first degree, and therefore it passes through a given curve.

324. If ten spheres, $\alpha_1, \alpha_2\ldots\alpha_{10}$, be all generators of the same cyclide W, their equations are connected by a linear relation,

$$l_1\alpha_1^2+l_2\alpha_2^2\ldots l_{10}\alpha_{10}^2=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (199)$$

Demonstration. Let $\alpha_1=\lambda_1\alpha+\mu_1\beta+\nu_1\gamma+\varrho_1\delta=0$, &c.; and writing down the conditions that α_1, α_2, &c. are generating spheres of W, and eliminating linearly the ten quantities

$$\alpha^2,\ \beta^2,\ \gamma^2,\ \delta^2,\ \alpha\beta,\ \alpha\gamma,\ \alpha\delta,\ \beta\gamma,\ \beta\delta,\ \gamma\delta,$$

we get the following determinant:—

$$\begin{vmatrix} \lambda_1^2 & \mu_1^2 & \nu_1^2 & \varrho_1^2 & \lambda_1\mu_1 & \lambda_1\nu_1 & \lambda_1\varrho_1 & \mu_1\nu_1 & \nu_1\varrho_1 & \varrho_1\mu_1 \\ \lambda_2^2 & \mu_2^2 & \nu_2^2 & \varrho_2^2 & \lambda_2\mu_2 & \lambda_2\nu_2 & \lambda_2\varrho_2 & \mu_2\nu_2 & \nu_2\varrho_2 & \varrho_2\mu_2 \\ 3 & , & , & , & , & , & , & , & , & , \\ 4 & , & , & , & , & , & , & , & , & , \\ 5 & , & , & , & , & , & , & , & , & , \\ 6 & , & , & , & , & , & , & , & , & , \\ 7 & , & , & , & , & , & , & , & , & , \\ 8 & , & , & , & , & , & , & , & , & , \\ 9 & , & , & , & , & , & , & , & , & , \\ 10 & , & , & , & , & , & , & , & , & , \end{vmatrix}=0. \quad . \quad . \quad . \quad . \quad (200)$$

the numbers 3, 4, 5, 6, 7, 8, 9, 10 in the determinant (200) should be small suffix numbers.

but this is also the condition that the squares should be connected by the linear relation.

325. Propositions similar to those of the three last articles hold for sphero-quartics and also for bicirculars. Thus the analogue of art. 322 is, being given five sphero-quartics S_1, S_2, S_3, S_4, S_5, and if multiples l_1, l_2 . . l_5 be determined so that $l_1S_1+l_2S_2+l_3S_3+l_4S_4+l_5S_5$ may be the square of a circle ν, then the envelope of ν is a sphero-quartic. The analogue of art. 324 is, if six circles be generators of the same sphero-quartic, their equations are connected by a linear relation.

326. To find the equation of the binodal cyclide formed by the generating spheres which touch W along the sphero-quartic in which W is intersected by the sphere $\lambda\alpha+\mu\beta+\nu\gamma+\varrho\delta$, where λ, μ, ν, ϱ are multiples.

The equation of any cyclide touching W along this curve will be of the form

$$kW+(\lambda\alpha+\mu\beta+\nu\gamma+\varrho\delta)^2,$$

and it is required to determine k so that this cyclide will be binodal. We find in this case Φ, Θ', Δ' all $=0$; the invariant equation has therefore three roots $=0$; and if we denote by σ the tangential expression $(A, B, C, D, L, M, N, P, Q, R)(\lambda, \mu, \nu, \varrho)^2$, the equation of the required binodal will be

$$\sigma W=\Delta(\lambda\alpha+\mu\beta+\nu\gamma+\varrho\delta)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (201)$$

Cor. 1. When $\lambda\alpha+\mu\beta+\nu\gamma+\varrho\delta$ is a generating sphere of W, the binodal (201) reduces to $\sigma=0$.

Cor. 2. If α', β', γ', δ' be the sphero-coordinates of the points polar to $\lambda\alpha+\mu\beta+\nu\gamma+\varrho\delta$ with respect to W, and if W′ be the result of substituting α', β', γ', δ' in W, then we have

$$\sigma=\Delta W'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (202)$$

For the binodal circumscribed to W, whose nodes are α', β', γ', δ', is

$$WW'=(\lambda\alpha+\mu\beta+\nu\gamma+\varrho\delta)^2$$

(see art. 280), and eliminating $\lambda\alpha+\mu\beta+\nu\gamma+\varrho\delta$ between this and (201), we get (202).

327. To find the condition that the circle of intersection of two spheres shall have double contact with W. Let W be given by the general equation, and let the spheres be $\lambda\alpha+\mu\beta+\nu\gamma+\varrho\delta$, $\lambda'\alpha+\mu'\beta+\nu'\gamma+\varrho'\delta$, then the required condition is the determinant

$$\begin{vmatrix} a, & n, & m, & p, & \lambda, & \lambda', \\ n, & b, & l, & q, & \mu, & \mu', \\ m, & l, & c, & r, & \nu, & \nu', \\ p, & q, & r, & d, & \varrho, & \varrho', \\ \lambda, & \mu, & \nu, & \varrho, & & \\ \lambda', & \mu', & \nu', & \varrho', & & \end{vmatrix}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (203)$$

328. The condition $\sigma=0$ (see art. 326), that the sphere $\lambda\alpha+\mu\beta+\nu\gamma+\varrho\delta$ should be a generator of W, is a contravariant of the third order in the coefficients of W. Hence, if

we substitute for each coefficient a, $a+ka'$, we get the condition that $\lambda\alpha+\mu\beta+\nu\gamma+\rho\delta$ shall be a generator of the cyclide $W+kW'$. The condition will be of the form

$$\sigma+k\tau+k^2\tau'+k^3\sigma'=0. \quad \ldots \ldots \ldots \quad (204)$$

In terms of the functions σ, τ, τ', σ' can be expressed the condition that the sphere $\lambda\alpha+\mu\beta+\nu\gamma+\rho\delta$ shall have any permanent relation with the cyclides W, W'; as, for instance, that it should intersect them in sphero-quartics w, w' connected by such permanent relations as can be expressed by relations between the coefficients of the discriminant of $w+kw'$. Thus if we form the discriminant with respect to k of equation (204), we get the condition that the sphero-quartics in which $\lambda\alpha+\mu\beta+\nu\gamma+\rho\delta$ intersects W and W' shall have double contact; in other words, the discriminant is the condition that the sphero-quartics shall have a common generating circle which touches both quartics at the same points. Again, $\tau=0$ is the condition that the sphero-quartics w and w' are so related that the harmonic circles (see 'Bicircular Quartics,' art. 184) of one are generators of the other.

329. The coefficients σ, τ, τ', σ' of equation (204), and the discriminant of the same equation, have another geometrical interpretation. Thus σ and σ' are the equations in tetrahedral coordinates of the focal quadric of W and W', λ', μ', ν, ρ being the current coordinates (see art. 27), and τ, τ' are quadrics covariant with σ, σ'. Thus τ "is the locus of a point whence cones circumscribing σ and σ' are so related that three edges of one can be found forming a self-conjugate system with regard to the second, and three tangent planes of the second which form a self-conjugate system with regard to the first" (see SALMON's 'Geometry of Three Dimensions,' page 159). The discriminant of (204) is the developable circumscribed to σ and σ'; in other words, the locus of the centre of $\lambda\alpha+\mu\beta+\nu\gamma+\rho\delta$ is the developable. Hence we infer:—*The locus of the centre of a variable sphere which cuts two cyclides in sphero-quartics having double contact is the developable circumscribed about the focal quadrics of the cyclides which correspond to their common inversion sphere.*

330. If we suppose the cyclide W' of the last article to become U^2, we have the following theorem:—*The locus of the centre of a sphere* S *intersecting the cyclide* W *in a sphero-quartic* WS *which has double contact with the sphero-quartic* WU *is the developable* Σ *formed by tangent planes to* U *along* WU (see Chapter VIII., art. 142).

331. If $W=a\alpha^2+b\beta^2+c\gamma^2+d\delta^2=0$, then $\sigma\equiv A\lambda^2+B\mu^2+C\nu^2+D\rho^2$, where $A=bcd$, $B=cda$, &c.; and changing a into $a+ka'$, &c., we get

$$\tau\equiv\left(\frac{b'}{b}+\frac{c'}{c}+\frac{d'}{d}\right)A\lambda^2+\left(\frac{c'}{c}+\frac{d'}{d}+\frac{a'}{a}\right)B\mu^2+\left(\frac{d'}{d}+\frac{a'}{a}+\frac{b'}{b}\right)C\nu^2+\left(\frac{a'}{a}+\frac{b'}{b}+\frac{c'}{c}\right)D\rho^2; \quad . \quad (205)$$

and the cyclide which has (205) for a focal quadric will be got by reciprocating (205) and substituting α, β, γ, δ for the variables. Hence the required cyclide will be

$$\frac{\alpha^2}{b'cd+bc'd+bcd'}+\frac{\beta^2}{c'db+cd'b+cdb'}+\frac{\gamma^2}{d'ab+da'b+dab'}+\frac{\delta^2}{a'bc+ab'c+abc'}. \quad . \quad . \quad (206)$$

This will be the locus of the nodes of binodals circumscribed to W W', *the same points*

being nodes for both, if three harmonic spheres of one binodal be three generating spheres of the other.

332. We can reciprocate the process of recent articles. Thus, let V, V′ be the focal quadrics of two cyclides W, W′ having a common inversion sphere, then $V+kV'$ will be the focal quadric of a cyclide whose equation we can find as follows, viz. form the tangential equation of $V+kV'$, and substitute α, β, γ, δ for the variables, the required equation will be

$$\frac{\alpha^2}{a^{-1}+ka'^{-1}}+\frac{\beta^2}{b^{-1}+kb'^{-1}}+\frac{\gamma^2}{c^{-1}+kc'^{-1}}+\frac{\delta^2}{d^{-1}+kd'^{-1}}=0; \quad . \quad . \quad . \quad (207)$$

a^{-1}, b^{-1}, &c. are evidently the coefficients of V, since a, b, &c. are the coefficients of W. The discriminant of (207) with respect to k will be the envelope of all the cyclides which it can represent by varying k, that is, it will be the *tore* with which they all have double contact. The curve of *taction* of any of them with it will be of the eighth degree, being the intersection of two cyclides of the fourth degree.

The geometrical interpretation of the discriminant is, that it is the envelope of a variable sphere cutting U *orthogonally, and whose centre moves along the twisted quartic* (V V′).

333. We can get the equation of the cuspidal edge as follows: differentiate (207) twice with respect to k, and we get a system reducible to the following equations:—

$$\frac{a^{-2}\alpha^2}{(a^{-1}+ka'^{-1})^3}+\frac{b^{-2}\beta^2}{(b^{-1}+kb'^{-1})^3}+\frac{c^{-2}\gamma^2}{(c^{-1}+kc'^{-1})^3}+\frac{d^{-2}\delta^2}{(d^{-1}+kd'^{-1})^3}=0. \quad . \quad . \quad . \quad (208)$$

$$\frac{(aa')^{-1}\alpha^2}{(a^{-1}+ka'^{-1})^3}+\frac{(bb')^{-1}\beta^2}{(b^{-1}+kb'^{-1})^3}+\frac{(cc')^{-1}\gamma^2}{(c^{-1}+kc'^{-1})^3}+\frac{(dd')^{-1}\delta^2}{(d^{-1}+kd'^{-1})^3}=0. \quad . \quad . \quad . \quad (209)$$

$$\frac{(a')^{-2}\alpha^2}{(a^{-1}+ka'^{-1})^3}+\frac{(b')^{-2}\beta^2}{(b^{-1}+kb'^{-1})^3}+\frac{(c')^{-2}\gamma^2}{(c^{-1}+kc'^{-1})^3}+\frac{(d')^{-2}\delta^2}{(d^{-1}+kd'^{-1})^3}=0. \quad . \quad . \quad . \quad (210)$$

The result of eliminating k between these equations will be a pair of equations representing two surfaces whose curve of intersection will be the cuspidal edge.

Now solving the equations (208), (209), (210), we get

$$\frac{\alpha^2}{(a^{-1}+ka'^{-1})^3}=\begin{vmatrix} \frac{1}{b^2}, & \frac{1}{c^2}, & \frac{1}{d^2}, \\ \frac{1}{bb'}, & \frac{1}{cc'}, & \frac{1}{dd'}, \\ \frac{1}{b'^2}, & \frac{1}{c'^2}, & \frac{1}{d'^2}, \end{vmatrix}=\text{A}^2 \text{ suppose.}$$

Hence $(a^{-1}+ka'^{-1})=\left(\frac{\alpha^2}{\text{A}^2}\right)^{\frac{1}{3}}$, with similar values for $(b^{-1}+kb'^{-1})$, &c.; and substituting in the equation (207), we get

$$(\text{A}^2\alpha^4)^{\frac{1}{3}}+(\text{B}^2\beta^4)^{\frac{1}{3}}+(\text{C}^2\gamma^4)^{\frac{1}{3}}+(\text{D}^2\delta^4)^{\frac{1}{3}}=0\,. \quad . \quad . \quad . \quad . \quad . \quad (211)$$

as another surface on which the cuspidal edge lies. But if we eliminate k between any three of the equations for $a^{-1}+ka'^{-1}$, $b^{-1}+kb'^{-1}$, &c., we get four equations of binodal

cyclides, each of the twelfth degree, on which the cuspidal edge also lies. These equations will be similar in form to the sextic cones containing the cuspidal edge of the developable circumscribed about two quadrics (see art. 182).

Cor. *If any of these binodals be inverted from one of its nodes, it becomes one of the sextic cones of* art. 182.

334. The equation (207), cleared of fractions, becomes

$$\Delta' W + kT + k^2 T' + k^3 \Delta W'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (212)$$

If in this equation we put $k = \frac{\Delta'}{\Delta}\lambda$, we get

$$\Delta^2 W + \lambda\Delta T + \lambda^2 \Delta' T' + \lambda^3 \Delta'^2 W' = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (213)$$

Compare SALMON'S 'Geometry of Three Dimensions,' art. 206. The value of T is

$$\Delta'\left\{a\left(\frac{b}{b'}+\frac{c}{c'}+\frac{d}{d'}\right)\alpha^2+b\left(\frac{c}{c'}+\frac{d}{d'}+\frac{a}{a'}\right)\beta^2+c\left(\frac{d}{d'}+\frac{a}{a'}+\frac{b}{b'}\right)\gamma^2+d\left(\frac{a}{a'}+\frac{b}{b'}+\frac{c}{c'}\right)\delta^2\right\}; \quad . \quad (214)$$

and T′ is got from T by interchanging accented and unaccented letters.

In terms of the cyclides T, T′ can be expressed all the cyclides having permanent relations to W, W′. Thus if

S be the reciprocal of W with respect to W′,

S′ be the reciprocal of W′ with respect to W,

then

$$T = \Theta' W - S', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (215)$$

$$T' = \Theta W' - S. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (216)$$

Hence W, S′, T have a common curve of intersection.

335. The discriminant of (212) is

$$27\Delta^2\Delta'^2 W^2 W'^2 + 4(\Delta' W T'^3 + \Delta W' T^3) - TT'(TT' + 18\Delta\Delta' WW'), \quad . \quad . \quad (217)$$

an equation of the sixteenth degree, since it contains α, β, γ, δ in the eighth degree. *The imaginary circle at infinity is on this surface a multiple curve of the eighth degree, so that it is an octavic cyclide.*

By making W = 0, we see that the surface touches W along the curve WT, and that it meets W again in the curve of intersection of W with $T'^2 - 4\Delta W' T$; this represents a system of eight circles which are generators of W. The sections of (217) by the spheres of reference are easily obtained; for, by a known process, the section of the discriminant of (207) by the sphere δ will be the sphero-quartic squared,

$$\left(\frac{aa'\alpha^2}{ad'-a'd}+\frac{bb'\beta^2}{bd'-b'd}+\frac{cc'\gamma^2}{cd'-c'd}\right)^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (218)$$

multiplied by the discriminant of

$$\frac{\alpha^2}{a^{-1}+ka'^{-1}}+\frac{\beta^2}{b^{-1}+kb'^{-1}}+\frac{\gamma^2}{c^{-1}+kc'^{-1}},$$

or the system of four circles,

$$\alpha\sqrt{aa'(bc'-b'c)}\pm\beta\sqrt{bb'(ca'-c'a)}\pm\gamma\sqrt{cc'(ab'-a'b)}=0. \quad . \quad . \quad . \quad (219)$$

The section is therefore a sphero-quartic counted twice and four circles.

Cor. *The four circles are generators of the sphero-quartic.*

336. We can show geometrically that a generating circle of the cyclide W on each of the eight generating spheres common to W, W′, S′, or to W, W′, T′, is also a generating circle of the cyclide (212), and therefore that these eight circles form the locus which is the intersection of W with $T'^2-4\Delta TW$ (see art. 335). Since S′ and W′ are reciprocals with respect to W, it is evident that, on the eight spheres which are common generators of W, W′, S′, at the points of contact of W, W′ with S′ these spheres are coincident. Hence one of the generating circles of W on each of these generating spheres is also a generating sphere of the cyclide (212); hence W and (212) have eight common generating circles.

337. The cyclide (212) is the same generalization of the developable circumscribed to two quadrics which a cyclide of the second degree in α, β, γ, δ is of a quadric,—thus to the generating lines of one corresponding generating circles of the other, and to the nodal conics corresponding nodal sphero-quartics, and so on. Hence, by the system of substitutions established in Chapter XIII., we can get from the properties proved in Chapter VIII. of the developable Σ, theorems which hold for the cyclide (212). The following are a few illustrations:—

1°. Eight lines of Σ meet any arbitrary line. Hence *eight generating circles of* (212) *meet any arbitrary circle orthogonal to the sphere* U.

2°. The curves of taction of Σ divide homographically the lines of the system. Hence *the curves of taction of* (212) *divide homographically the circles of the system.*

3°. The nodal lines of Σ divide equianharmonically the lines of the system. Hence *the nodal sphero-quartics of* (212) *divide equianharmonically the circles of the system.*

4°. Any line of Σ meet its curves of taction in points the tangents at which to the curves of taction envelope a plane conic.

Hence *any generating circle of* (212) *meet its curves of taction in points, the generating circles of the curves of taction through which are generators of a sphero-quartic.*

338. Since the surface (212) is the envelope of a variable sphere cutting U orthogonally, and whose centre moves along the twisted quartic (VV′), then (VV′) is the déférente. From this generation we can also infer the properties of (212). *Thus the cuspidal edge of* (212) *is the locus of the limiting points composed of the sphere* U *and the osculating planes of* (VV′).

2°. *There are sixteen pairs of stationary points on the cuspidal edge; these correspond to the stationary planes of* (VV′).

3°. *Any sphere cutting* U *orthogonally meets the cuspidal edge in twelve pairs of inverse points.* This follows from $n=12$ (see art. 224).

4°. *The cuspidal edge is an anallagmatic,* U *being the sphere of inversion.*

339. To find the locus of a pair of inverse points whose polar spheres with respect

to W will be a generating sphere of $W+kW'$. We have then in $\sigma+k\tau+k^2\tau'+k^3\sigma'$ to substitute $\frac{dW}{d\alpha}$, $\frac{dW}{d\beta}$, $\frac{dW}{d\gamma}$, $\frac{dW}{d\delta}$ for λ, μ, ν, ϱ; the result is expressible in terms of the covariants; it is

$$\Delta W+k(\Theta W-\Delta W')+k^2(\Phi W-T')+k^3(\Theta' W-T)=0. \quad . \quad . \quad . \quad . \quad (220)$$

In like manner the locus of inverse pairs of points whose polar spheres with respect to W' are generating spheres of $W+kW'$ is

$$\Theta W-T'+k(\Phi W'-T)+k^2(\Theta W'-\Delta' W)+k^3\Delta' W'=0. \quad . \quad . \quad . \quad . \quad (221)$$

340. To find the locus of a pair of inverse points whose polar spheres with respect to W, W' form a conjugate system with respect to $W+kW'$. Let

$$W=a\alpha^2+b\beta^2+c\gamma^2+d\delta^2, \quad W'=a'\alpha^2+b'\beta^2+c'\gamma^2+d'\delta^2,$$

and the locus will be

$$\begin{vmatrix} a+ka', & 0, & 0, & 0, & a\alpha, \\ 0, & b+kb', & 0, & 0, & b\beta, \\ 0, & 0, & c+kc', & 0, & c\gamma, \\ 0, & 0, & 0, & d+kd', & d\delta, \\ a\alpha, & b\beta, & c\gamma, & d\delta, & 0, \end{vmatrix}=0. \quad . \quad . \quad (222)$$

This can be expressed in terms of the covariants, and is the cyclide

$$\Delta W'+kT'+k^2T'+k^3\Delta' W=0 \text{ (compare 212)}. \quad . \quad . \quad . \quad . \quad . \quad (223)$$

(341). To find the condition that a given circle should have a given pair of inverse points common to the curve of intersection of two given cyclides W, W'. Suppose we have formed the condition (see art. 327) $R=0$, that the given circle should have double contact with W, and that we substitute in it for each coefficient a, $a+ka'$, &c., the condition becomes

$$R+k\pi+k^2R'=0 \; ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (224)$$

and if the given circle has any arbitrary position, we can, by solving this quadratic for k, determine two cyclides through the intersection of W and W', each having double contact with the given circle; but if the given circle has a pair of inverse points in common with the curve (WW'), the cyclides having double contact which can be drawn through (WW') become coincident, and the equation (224) becomes a perfect square. Hence the required condition is the discriminant

$$4RR'-\pi^2=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (225)$$

Cor. $\pi=0$ is the condition that the pair of segments which W intercepts on the given circle should be harmonic conjugates to those which W' intercepts on it.

342. If $W=a\alpha^2+b\beta^2+c\gamma^2+d\delta^2=0$, $W'=a'\alpha^2+b'\beta^2+c'\gamma^2+d'\delta^2=0$, and if the circle be the intersection of the spheres

$$\lambda\alpha+\mu\beta+\nu\gamma+\varrho\delta=0, \quad \lambda'\alpha+\mu'\beta+\nu'\gamma+\varrho'\delta=0,$$

then

$$R = \text{sum of the six terms } \Sigma ab(\nu\varrho'-\nu'\varrho)^2,$$

$$\pi = \text{the sum } \Sigma(ab'+a'b)(\nu\varrho'-\nu'\varrho)^2.$$

Therefore

$$\left.\begin{aligned}\pi^2-4RR' &= \Sigma(ab'-a'b)^2(\nu\varrho'-\nu'\varrho)^4\\ &+2(ab'-a'b)(ac'-a'c)(\mu\varrho'-\mu'\varrho)^2(\nu\varrho'-\nu'\varrho)^2\\ &+2\Sigma\{(ad'-a'd)(cb'-c'b)+(ac'-a'c)(db'-d'b)\}\\ &\times(\lambda\mu'-\lambda'\mu)^2(\nu\varrho'-\nu'\varrho)^2=0.\end{aligned}\right\} \quad . \quad . \quad . \quad (226)$$

343. *The surface generated by the circles which have double contact with the curve of intersection of* W *and* W' *may be generated as the envelope of a variable sphere which cuts* U *orthogonally and whose centre moves along the cuspidal edge of the developable circumscribed about the focal quadrics of* W, W' *which correspond to their common sphere of inversion* U. For let us consider any circle having double contact with the curve (WW'). Then, since a circle having double contact with WW' is orthogonal to the sphere U, it is plain that a line through its centre perpendicular to its plane is a line of the developable circumscribed to the two focal quadrics, and therefore the sphere containing two consecutive circles will have its centre at the point of intersection of two consecutive lines of the developable that is on the cuspidal edge. Hence the theorem is proved.

344. *The curve* (W W') *is a cuspidal edge on the surface generated by the circles having double contact with* (W W'). This is evident; for any circle having double contact is the characteristic of the surface (see MONGE, 'Application de l'Analyse à la Géométrie,' p. 53), and the points of intersection of each characteristic with the consecutive one form the cuspidal edge. Hence the proposition is proved.

Cor. The cuspidal edge is an anallagmatic.

345. *To find the equation of the surface generated by the circles which have double contact with the curve* (W W').

Let us consider any pair of inverse points on any circle which has double contact with (W W'). The polar sphere of this pair, with respect either to W or W', passes evidently through the two points of contact of the circle under consideration with the curve (W W'). The circle of intersection, therefore, of the two polar spheres intersects the curve (W W') in two points; therefore the equation of the required surface is found by substituting in the equation (226) for λ, μ, ν, ϱ, $\frac{dW}{d\alpha}$, $\frac{dW}{d\beta}$, $\frac{dW}{d\gamma}$, $\frac{dW}{d\delta}$, and for λ', μ', ν', ϱ', $\frac{dW'}{d\alpha}$, $\frac{dW'}{d\beta}$, $\frac{dW'}{d\gamma}$, $\frac{dW'}{d\delta}$.

This surface is of the sixteenth degree, being of the eighth degree in $\alpha, \beta, \gamma, \delta$; when we use the canonical forms $a\alpha^2+b\beta^2+c\gamma^2+d\delta^2$, $a'\alpha^2+b'\beta^2+c'\gamma^2+d'\delta^2$ for W, W', the

equation of the surface becomes

$$\left.\begin{array}{l}\Sigma(ab'-a'b)^4(cd'-c'd)^4\gamma^4\delta^4\\ \quad+2\Sigma(ab'-a'b)(ac'-a'c)(cd'-c'd)^2(bd'-b'd)^2\beta^2\gamma^2\delta^4\\ \quad+2\alpha^2\beta^2\gamma^2\delta^2\{(ab'-a'b)(cd'-c'd)-(ad'-a'd)(bc'-b'c)\}\\ \quad\times\{(ad'-a'd)(bc'-b'c)-(bd'-b'd)(ca'-c'a)\}\\ \quad\times\{(bd'-b'd)(ca'-c'a)-(ab'-a'b)(cd'-c'd)\}.\end{array}\right\} \quad . \quad . \quad (227)$$

The imaginary circle at infinity is a multiple curve of the order eight on this surface.

Cor. 1. When we make $\delta=0$ in equation (227) we get a perfect square. Hence each of the four spheres of reference meets the surface in a double line on the surface. These double lines correspond to the double lines of the developable Δ (see Chapter VIII.), and each of them has six double points. Thus the sphero-octavic in which δ intersects the surface is expressed in terms of α, β, γ, and is of the form

$$\frac{(bc'-b'c)^2}{bc\alpha^2}+\frac{(ca'-c'a)^2}{c'a'\beta^2}+\frac{(ab'-a'b)^2}{a'b'\gamma^2}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (228)$$

Hence the three pairs of points $(\alpha\beta)$, $(\beta\gamma)$, $(\gamma\alpha)$ are double points.

Cor. 2. The equation (227), expressed in terms of the covariant cyclides, is given by the determinant

$$\begin{vmatrix} 2(\Theta WW'-T'W-\Delta W'^2), & \Phi WW'-TW-T'W', \\ \Phi WW'-TW-T'W', & 2(\Theta' WW'-TW'-\Delta' W^2), \end{vmatrix}=0. \quad . \quad (229)$$

Cor. 3. The surface also meets W in the curve of intersection of W with $T'^2-4\Delta TW'$, which we have shown represents a system of eight circles which are generators of W.

Cor. 4. Any arbitrary circle orthogonal to U meets eight generating circles of the surface, and the spheres determined by the arbitrary circle and the eight meeting circles are generators of a cyclide.

346. If a cyclide W be given by the equation

$$a\alpha^2+b\beta^2+c\gamma^2+d\delta^2=0,$$

and also by this other equation referred to different spheres,

$$a'\alpha'^2+b'\beta'^2+c'\gamma'^2+d'\delta'^2=0,$$

then we can infer, as in SALMON's 'Geometry of Three Dimensions,' art. 192, the following theorems:—

1°. The eight spheres α, β, γ, δ, α', β', γ', δ' are generators of the same cyclide.

2°. The two quartets of pairs of inverse points

$$(\alpha\ \beta\ \gamma),\quad(\alpha\ \beta\ \delta),\quad(\alpha\ \gamma\ \delta),\quad(\beta\ \gamma\ \delta),$$
$$(\alpha'\ \beta'\ \gamma'),\quad(\alpha'\ \beta'\ \delta'),\quad(\alpha'\ \gamma'\ \delta'),\quad(\beta'\ \gamma'\ \delta'),$$

lie on a cyclide; this theorem (2°) may be inferred by reciprocation from 1°.

347. If W, W′ be two cyclides, $S=\lambda\alpha+\mu\beta+\nu\gamma+\varrho\delta=0$ any sphere, and it be required to find the condition that the binodal cyclides formed by the generating spheres which touch W, W′ along their curves of intersection with S may intersect along a third cyclide W″, the three cyclides being given by the equations

$$W=a\alpha^2+b\beta^2+c\gamma^2+d\delta^2=0,$$
$$W'=a'\alpha^2+b'\beta^2+c'\gamma^2+d'\delta^2=0,$$
$$W''=a''\alpha^2+b''\beta^2+c''\gamma^2+d''\delta^2=0,$$

then we have

$$\sigma=bcd\lambda^2+cda\mu^2+dab\nu^2+abc\varrho^2, \quad \Delta=abcd,$$
$$\sigma'=b'c'd'\lambda^2+c'da'\mu^2+d'a'b'\nu^2+a'b'c\varrho^2, \quad \Delta'=a'b'c'd'.$$

Hence (see art. 326) the equations of the binodals which circumscribe W and W′ along the sphero-quartics (WS) and (W′S) are

$$\left(\frac{\lambda^2}{a}+\frac{\mu^2}{b}+\frac{\nu^2}{c}+\frac{\varrho^2}{d}\right)W-S^2=0,$$
$$\left(\frac{\lambda^2}{a'}+\frac{\mu^2}{b'}+\frac{\nu^2}{c'}+\frac{\varrho^2}{d'}\right)W'-S^2=0.$$

Hence we must have, by the given conditions,

$$\left(\frac{\lambda^2}{a}+\frac{\mu^2}{b}+\frac{\nu^2}{c}+\frac{\varrho^2}{d}\right)W-\left(\frac{\lambda^2}{a'}+\frac{\mu^2}{b'}+\frac{\nu^2}{c'}+\frac{\varrho^2}{d'}\right)W'\equiv kW'',$$

where k is some constant; and equating coefficients, we get

$$\mu^2\left(\frac{a}{b}-\frac{a'}{b'}\right)+\nu^2\left(\frac{c}{d}-\frac{c'}{d'}\right)+\varrho^2\left(\frac{a}{d}-\frac{a'}{d'}\right)=ka'',$$
$$\nu^2\left(\frac{b}{c}-\frac{b}{c'}\right)+\varrho^2\left(\frac{b}{d}-\frac{b'}{d'}\right)+\lambda^2\left(\frac{b}{a}-\frac{b'}{a'}\right)=kb'',$$
$$\varrho^2\left(\frac{c}{d}-\frac{c'}{d'}\right)+\lambda^2\left(\frac{c}{a}-\frac{c'}{a'}\right)+\mu^2\left(\frac{c}{b}-\frac{c'}{b'}\right)=kc'',$$
$$\lambda^2\left(\frac{d}{a}-\frac{d'}{a'}\right)+\mu^2\left(\frac{d}{b}-\frac{d'}{b'}\right)+\nu^2\left(\frac{d}{c}-\frac{d'}{c'}\right)=kd'';$$

multiplying these equations by $\frac{\lambda^2}{aa'}$, $\frac{\mu^2}{bb'}$, $\frac{\nu^2}{cc'}$, $\frac{\varrho^2}{dd'}$, and adding, the left side vanishes identically. Hence the required condition is

$$\frac{\lambda^2a''}{aa'}+\frac{\mu^2b''}{bb'}+\frac{\nu^2c''}{cc'}+\frac{\varrho^2d''}{dd'}=0. \quad \ldots \ldots \ldots \quad (230)$$

348. The envelope of the sphere $S=\lambda\alpha+\mu\beta+\nu\gamma+\varrho\delta=0$ is the cyclide

$$\frac{aa'}{a''}\alpha^2+\frac{bb'}{b''}\beta^2+\frac{cc'}{c''}\gamma^2+\frac{dd'}{d''}\delta^2=0, \quad \ldots \ldots \ldots \quad (231)$$

the tangential equation of its focal quadric being (230); I shall denote the cyclide (231) by W''_1.

349. *If the cyclides* W, W′, W″ *pass through a common curve, the cyclides* W, W′, W''_1 *are inscribed in the same cyclic developable* (see art. 335).

Demonstration. The focal quadrics of W, W′, W''_1 are

$$F=\frac{\lambda^2}{a}+\frac{\mu^2}{b}+\frac{\nu^2}{c}+\frac{\varrho^2}{d},$$

$$F'=\frac{\lambda^2}{a'}+\frac{\mu^2}{b'}+\frac{\nu^2}{c'}+\frac{\varrho^2}{d'},$$

$$F''_1=\frac{a''\lambda^2}{aa'}+\frac{b''\mu^2}{bb'}+\frac{c''\nu^2}{cc'}+\frac{d''\varrho^2}{dd'}.$$

Now if $W''=W+kW'$, we must have $a''=a+ka'$ &c. Hence $F''_1=F+kF'$, that is, the focal quadrics pass through a common curve. Hence the proposition is proved.

350. *If the cyclide* W″ *pass through the curve* (W W′), *the cyclide* W''_1 *is inscribed in six cyclides passing through* (W W′).

Demonstration. If two cyclides, Ω, Ω′, be such that one is inscribed in the other, then the reciprocal of Ω with respect to Ω′ is inscribed in Ω′, and also the reciprocal of Ω′ with respect to Ω.

Let $\Omega=A\alpha^2+B\beta^2+C\gamma^2+D\delta^2$, $\Omega'=A'\alpha^2+B'\beta^2+C'\gamma^2+D'\delta^2$. Hence the required condition will be the determinant

$$\begin{vmatrix} A, & B, & C, & D, \\ A', & B', & C', & D', \\ \frac{A^2}{A'}, & \frac{B^2}{B'}, & \frac{C^2}{C'}, & \frac{D^2}{D'}, \\ \frac{A'^2}{A}, & \frac{B'^2}{B}, & \frac{C'^2}{C}, & \frac{D'^2}{D}, \end{vmatrix}=0,$$

or $\frac{1}{ABCD}\cdot\frac{1}{A'B'C'D'}(AB'-A'B)(AC'-A'C)(AD'-A'D)(BC'-B'C)(CD'-C'D)(BD'-B'D)=0.$

Hence $(AB'-A'B)(AC'-A'C)(AD'-A'D)(BC'-B'C)(CD'-C'D)(BD'-B'D)=0$ (232)

is the condition of Ω being inscribed in Ω′, that is, the product of the six determinants of the matrix

$$\begin{Vmatrix} A, & B, & C, & D, \\ A', & B', & C', & D', \end{Vmatrix}$$

as is otherwise evident.

To apply the condition (232), we have

$$W''_1=\frac{aa'}{a+ka'}\alpha^2+\frac{bb'}{b+kb'}\beta^2+\frac{cc'}{c+kc'}\gamma^2+\frac{dd'}{d+ka'}\delta^2=0\ ;$$

and let the cyclide through the curve (W W′) be

$$(a+k'a')\alpha^2+(b+k'b')\beta^2+(c+k'c')\gamma^2+(d+k'd')\delta^2=0,$$

then we get, after some reduction, the condition of W''_1 being inscribed in $W+k'W'$

given by the equation

$$\left(kk'-\frac{a'b'}{ab}\right)\left(kk'-\frac{a'c'}{ac}\right)\left(kk'-\frac{a'd'}{ad}\right)\left(kk'-\frac{b'c'}{bc}\right)\left(kk'-\frac{b'd'}{bd}\right)\left(kk'-\frac{c'd'}{cd}\right)=0, \quad . \quad (233)$$

a sextic in k'. Hence the proposition is proved.

351. *Given four cyclides* W, W′, W″, W‴, *required to find the locus of a pair of inverse points such that their polar spheres, with respect to the four cyclides, may pass through the same pair of inverse points.*

The required locus is the Jacobian of the four cyclides

$$\begin{vmatrix} \frac{dW}{d\alpha}, & \frac{dW}{d\beta}, & \frac{dW}{d\gamma}, & \frac{dW}{d\delta}, \\ \frac{dW'}{d\alpha}, & \frac{dW'}{d\beta}, & \frac{dW'}{d\gamma}, & \frac{dW'}{d\delta}, \\ \frac{dW''}{d\alpha}, & \frac{dW''}{d\beta}, & \frac{dW''}{d\gamma}, & \frac{dW''}{d\delta}, \\ \frac{dW'''}{d\alpha}, & \frac{dW'''}{d\beta}, & \frac{dW'''}{d\gamma}, & \frac{dW'''}{d\delta}, \end{vmatrix} =0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (234)$$

Cor. *The envelope of a sphere whose pole points with respect to four cyclides are homospheric is the Jacobian of the four cyclides.*

352. The Jacobian is the locus of the nodes of all binodals which can be represented by $kW+k'W'+k''W''+k'''W'''$. Thus, there being given six pairs of inverse points, the locus of the nodes of all binodes which can pass through them is an anallagmatic surface of the eighth degree. For if W, W′, W″, W‴ be any cyclides through them, every cyclide through them can be represented by $kW+k'W'+k''W''+k'''W'''$, since this last form contains three independent constants, which are necessary to complete the solution.

Cor. 1. If in any case $kW+k'W'+k''W''+k'''W'''$ can represent two spheres, the intersection of these spheres is a circle on the Jacobian.

Cor. 2. If one of the cyclides W be a perfect square L^2, the Jacobian consists of a sphere and an anallagmatic surface of the third order in $\alpha, \beta, \gamma, \delta$, that is, a surface whose déférente is a surface of the third class.

Cor. 3. If the cyclides have in common four pairs of inverse points which are homospheric, the sphere through these points is a part of the Jacobian.

Cor. 4. If the four cyclides have a sphero-quartic curve common to all, the sphere through the sphero-quartic counts doubly in the Jacobian, which therefore reduces to a cyclide and the square of the sphere.

Cor. 5. The Jacobian of four Cartesian cyclides is a Cartesian cyclide.

353. If F, F′, F″, F‴ be the focal quadrics of W, W′, W″, W‴ in tangential coordinates, the déférente of the Jacobian of W . . . W‴ is the Jacobian in tangential coordinates of

$F \ldots F'''$, that is, the determinant

$$\begin{vmatrix} \frac{dF}{d\lambda}, & \frac{dF}{d\mu}, & \frac{dF}{d\nu}, & \frac{dF}{d\varrho}, \\ \frac{dF'}{d\lambda}, & \frac{dF'}{d\mu}, & \frac{dF'}{d\nu}, & \frac{dF'}{d\varrho}, \\ \frac{dF''}{d\lambda}, & \frac{dF''}{d\mu}, & \frac{dF''}{d\nu}, & \frac{dF''}{d\varrho}, \\ \frac{dF'''}{d\lambda}, & \frac{dF'''}{d\mu}, & \frac{dF'''}{d\nu}, & \frac{dF'''}{d\varrho}, \end{vmatrix} = 0 \quad \ldots \ldots \ldots \quad (235)$$

354. If a cyclide of the systems $kW+k'W'+k''W''$ have double contact with W''', the points of contact are evidently points on the Jacobian, and therefore lie somewhere on the curve of intersection of W''' with the Jacobian. Again, if a cyclide of the system $kW+k'W'$ have double contact with the curve $(W''\ W''')$, that is to say, if at one of the pairs of inverse points where $kW+'kW'$ meets W'' and W''' the generating spheres of $(kW+k'W')$, W'', W''' intersect in the same circle, the pair of points is evidently a pair of points on the Jacobian. It follows then that sixteen surfaces of the system $kW+k'W'$ can be described to have double contact with the curve $(W''\ W''')$, since the Jacobian is of the fourth degree in $(\alpha, \beta, \gamma, \delta)$, and each of the cyclides W'', W''' of the second degree, and each system of common values of α, β, γ, δ gives a pair of inverse points.

355. *Given three cyclides* W, W', W'', *the locus of a pair of inverse points whose polar spheres with respect to* W, W', W'' *have a common circle of intersection is the curve of the twelfth degree, which is common to the system of determinants*

$$\begin{Vmatrix} \frac{dW}{d\alpha}, & \frac{dW}{d\beta}, & \frac{dW}{d\gamma}, & \frac{dW}{d\delta}, \\ \frac{dW'}{d\alpha}, & \frac{dW'}{d\beta}, & \frac{dW'}{d\gamma}, & \frac{dW'}{d\delta}, \\ \frac{dW''}{d\alpha}, & \frac{dW''}{d\beta}, & \frac{dW''}{d\gamma}, & \frac{dW}{d\delta}. \end{Vmatrix} \quad \ldots \ldots \ldots \quad (236)$$

356. To find the condition in the invariants that two cyclides W, W' shall be so related that four generating spheres α, β, γ, δ of W' shall have the circles $(\alpha\beta)$, $(\alpha\gamma)$, $(\delta\beta)$, $(\delta\gamma)$ lying on W.

The equation of W must be of the form $L\beta\gamma+P\alpha\delta=0$, and the coefficients a, b, c, d must be wanting in the general equation of W'. Hence we have

$$\Delta=L^2P^2, \qquad \Theta=2LP(Lp+Pl),$$

$$\Phi=(Lp+Pl)^2+2LP(lp-mq-nr),$$

$$\Theta'=2(lp-mq-nr)(Lp+Pl).$$

Hence the required condition is

$$4\Delta\Theta\Phi=\Theta^3+8\Delta^2\Theta' . \quad \ldots \ldots \ldots \ldots \quad (237)$$

Cor. The condition that W shall have the circles $(\alpha\beta)$, $(\alpha\gamma)$, $(\delta\beta)$, $(\delta\gamma)$ lying on its surface, while the four pairs of inverse points $(\alpha\beta\gamma)$, $(\alpha\beta\delta)$, $(\alpha\gamma\delta)$, $(\beta\gamma\delta)$ shall lie on the surface of W′, is the equation, reciprocal of the former,

$$4\Delta'\Theta'\Phi=\Theta'^3+8\Delta'^2\Theta. \qquad (238)$$

357. We have seen if two cyclides W, W′ be reciprocals with respect to

$$U^2=\alpha^2+\beta^2+\gamma^2+\delta^2,$$

that their focal quadrics are reciprocals with respect to the sphere U. Hence it follows that if the focal quadric of W be a plane conic, the focal quadric of W′ will be a cone. Hence we have the following theorem:—*The reciprocal of a binodal cyclide is a sphero-quartic, and* vice versâ.

358. *If a cyclide* W *breaks up into two spheres, its reciprocal* W′ *breaks up into two spheres.* For if W breaks up into two spheres, its focal quadric is a quadric of revolution circumscribed to U, and the reciprocal of the focal quadric with respect to U is another quadric of revolution circumscribed to U.

This and the last article belong to the Chapter on reciprocation, but were accidentally omitted.

359. If we form the discriminant of $k\text{W}+k'\text{W}'+k''\text{W}''$, the coefficients of the several powers of k, k', k'' will be invariants of the system of cyclides. There are two invariants, however, of the system $k\text{W}+k'\text{W}'+k''\text{W}''$ which, as being combinants, deserve attention. These invariants we shall call I and J.

The combinant I *vanishes whenever any four of the eight generating spheres common to* W, W′, W″ *are connected by a linear relation, that is, pass through the same two points.*

It is easy to see that this is equivalent to the statement that I vanishes for the values of k, k', k'' which will make $k\text{W}+k'\text{W}'+k''\text{W}''$ represent two spheres. The equations of W, W′, W″, as having a common sphere of inversion, may plainly be written in the forms

$$\begin{aligned}\text{W}\;&=a\alpha^2+b\beta^2+c\gamma^2+d\delta^2+e\varepsilon^2,\\ \text{W}'\,&=a'\alpha^2+b'\beta^2+c'\gamma^2+d'\delta^2+e'\varepsilon^2,\\ \text{W}''&=a''\alpha^2+b''\beta^2+c''\gamma^2+d''\delta^2+e''\varepsilon^2,\end{aligned}$$

where $\alpha^2+\beta^2+\gamma^2+\delta^2+\varepsilon^2=0$ identically; and it is clear that I is the product of the ten determinants (a, b', c''), &c. For $(a, b', c'')\alpha^2+(d, b', c'')\delta^2+(e, b', c'')\varepsilon^2$ is evidently a cyclide of the system $k\text{W}+k'\text{W}'+k''\text{W}''$; and this reduces to two spheres if one of the determinants $(a\,b'\,c'')$ vanishes. Hence I is the product of the ten determinants.

Cor. *The combinant* I *vanishes also whenever any four of the eight pairs of inverse points common to* W, W′, W″ *are homospheric.*

360. *The combinant* J *vanishes whenever any two of the eight generating spheres common to* W, W′, W″ *are coincident; or again, when any two pairs of the eight pairs of common inverse points are coincident, so that* J *will be the tact-invariant of the three cyclides.*

If the generating sphere at a pair of inverse points common to the three surfaces pass

through a common circle, the consecutive pair of inverse points on this circle will be common to all the surfaces; such a pair of inverse points will be the two conic nodes of all binodal cyclides of the system $kW+k'W'+k''W''$.

For let the generating spheres at the given pair of inverse points be α, β, and $(a\alpha+b\beta)$, and the equations of the cyclides may be written $\alpha\delta+w_2$, $\beta\delta+w'_2$, and $(a\alpha+b\beta)\delta+w''_2$, where w_2, w'_2, w''_2 denote homogeneous functions of the second degree in α, β, γ; and it is evident that $aW+bW'-W''$ is a binodal cyclide having the given pair of points as conic nodes.

Cor. J will be of the sixteenth degree in the coefficients of W, W', W''. For if in J we substitute for each coefficient a of W, $a+ka_1$, where a_1 is the corresponding coefficient of a fourth cyclide W_1, it is evident that the degree of the result in k is the same as the number of cyclides of the system $W+kW_1$ which can be drawn to have double contact with the curve of intersections of the cyclides W' and W'', and the degree is therefore sixteen (see art. 354).